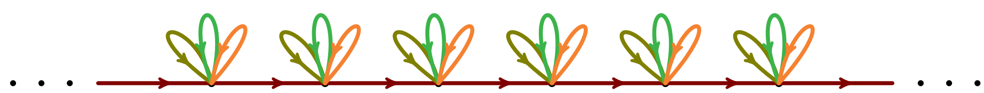
Math 490. ***Constructive geometry: from grounds to groups.*** **Fall 2025. Igor Mineyev.**
Topics, homework, and fun. Also pictures – scroll down to see them!

ground /graʊnd/ or /gruːnd/ *(n.)*

- The solid surface of the earth; a basis or foundation for reasoning, belief, or action; or the lower part or bottom of something.
- From Old English *grund*, meaning "bottom, foundation, land." Of Germanic origin; related to Dutch *grond* and German *Grund*, both of which retain a long [u]-like vowel in modern pronunciation. In early English, the vowel in "ground" was likewise [uː], but it shifted to [aʊ] between roughly 1400 and 1600 in southern England as part of wider native vowel changes.
- /graʊnd/: the modern pronunciation.
- /gruːnd/ (as in *group*): a more consistent pronunciation reflecting the word's etymology.



This course presents a highly unconventional, unified – and better – introduction to group theory, algebraic topology, graphs, etc, with an emphasis on explicit, geometric, constructive, algorithmic methods. I define what I call *grounds*; they are particular highly symmetric colored graphs that will serve as geometric *precursors* to groups. It will eventually happen that grounds are isomorphic to Cayley graphs of groups, but unlike Cayley graphs grounds are defined *before* any group is given or even defined, and their definitions are geometric and constructive. Grounds *lead* to groups rather than result from them.

I would claim that, conceptually speaking, grounds (that are constructive, highly symmetric colored graphs) are primary and groups (that are abstract, non-symmetric sets with abstract operation) are secondary. In this sense, *all* group theory is *geometric group theory*.

Conventionally, before students are ever told about geometric group theory, they are required to take a course in group theory/algebra. And to truly understand concepts used in geometric group theory, a course in algebraic topology might also be helpful (but certainly not necessary). This 490 course can be viewed as a beginner's introduction to both subjects, with no prior knowledge of group theory or algebraic topology required, and in a conceptual way that has not been done before – constructive-geometric and from the *ground* up.

Below, "$*$" means "turn in", "no $*$" means "do not turn in, but know how to solve". If a text is in the yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked "**for extra fun**" are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

Week 1. Integers and the +1 transformation, the associated graph. States, transformations = symmetries. Symmetries of the square. Symmetries of the 3-dimensional cube. Generating transformations. The state-symmetry graph.

Two ways to construct the (generalized) ground associated to an object and its symmetries (describe all states at once or start with one an keep applying the symmetries). The formal definition of a generating set of transformations. $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}_n$.

A graph, an edge-path = path.

Week 2. (Two classes.) A reduced path. A reduced loop. An abstract reduced path. An abstract reduced loop. An abstract cyclically reduced loop.

Why not use groups: they are black boxes. An informal idea of a ground. ~~Two main sources of groups.~~ Two main *algorithmic* sources of grounds (explicitly and algorithmically): state-symmetry graphs and gluing pieces. Generalized algorithms. Natural colorings of a state symmetry graph.

Week 3. A graph. Convention: orientations of edges are fixed and "opposite edges" do not exist. (This is different and more convenient notation from Serre's convention often used in algebra.) A map of graphs. Constructing highly symmetric colored graphs out of abstract loops or polygons: 1-dimensional gluing and 2-dimensional gluing. Folding "along the flow". Formal gluing principles. To fold or not to fold? Creating fixed points in the 2-complex. Guaranteeing homogeneity of the limit: first in, first out.

Fixed points of an object (under some set of transformations). The combinatorial structure of a polygon. To create a highly homogeneous colored graph by gluing abstract loops, is it enough to define the limiting graph as the union $\mathcal{G} := \bigcup_i \mathcal{G}_i$? Equivalence relations and quotient maps.

Homework 1. Hard copy due at 2pm on Friday, September 12, 2025.

(0) Note on the homework. Discussions are important part of any scientific research. *At the stage of solving* the homework problems, collaboration is allowed and even encouraged *if* you actively participate in solving the problem, not just copying a solution. *At the stage of writing out* the homework, do it on your own, write how you, yourself, would like to present a solution.

(1*) Investigate the set of states of the fidget-fiddle. Or the twiddle-fiddle-riddle, whichever name is more appropriate. (This set might not be unique, depending on your definition of "a state".) Then describe a natural generating set of transformations of the fidget-fiddle, or the twiddle-fiddle-riddle. (Again, there might be some choice involved.) Make at least two different choices for what can be meant by states and transformations, and draw the corresponding at least two colored graphs. Make it so that edges are colored, and multiple edges (naturally) have the same color (because they correspond to "similar" transformations, just applied to different states).

For extra fun.

(1) Take any set X and consider (formally) it as a set of states. Formally associate a transformation to each pair $(x, y) \in X \times X$, and draw, formally, an edge from x to y for each such pair (x, y). The result is what I called *the symmetric join of X* in the paper "Flows and joins of metric spaces" available on the website. This also

happens to be an example of a ground. If X happens to be a topological space, there is a natural topology on its symmetric join. If X happens to be a metric space, there is also a natural metric on its symmetric join. If $X$is, say, a Lie group, then its symmetric join can be interpreted as a ground: its edges can be colored in a certain way consistent with the definition of a ground that will be given later. If X is just a set, can the symmetric join be interpreted as a ground? Investigate. If yes, this will give another example of a ground, not coming from an object and its symmetries, and not coming from groups.

Week 4. The equivalence relation generated by $R \subseteq X \times X$, $R \mapsto \hat{R}$. The process of coarsening equivalence relations. $\mathcal{R}_i$, $\sim_i$ on X. Evenness of valence under the gluing procedure. Soccer balls and platonic solids. 1-skeleton. Two ways to realize spherical (planar) graphs as grounds: as symmetric colored graphs (symmetric formal state-symmetry graphs) and as obtained by gluing.

Is the result of gluing finite or infinite? Relation to probability theory: random choice of polygons with directions and colors. Equivalence relations and quotient maps.

The formal and precise definition of limit of the gluing procedure. When the quotient of a graph by an equivalence relation is again a graph? Compatibility conditions.

Week 5. Constructing graphs corresponding to oriented surfaces: closed oriented surfaces.

An informal description of small cancellation conditions, for graphs built out of polygons. Graphs corresponding to closed unoriented surfaces. Upstairs (free gluing of loops/ polygons) and downstairs (full gluing of polygons). Upstairs and downstairs for any (starting) family of polygons $\mathcal{Q} = \{Q_i \mid i \in I\}$.

Abstract edges. The gluing construction using abstract edges only. *The free ground* over a set of colors S, $\underline{F}(S)$. The definition of a tree in the setting of directed graphs.

For extra fun.

- Soccer balls or other interesting symmetric objects with interesting state-symmetry graphs. Examples of symmetric colorings of the 1-skeleton of the soccer ball. Examples of symmetric realizations of the 1-skeleton of the soccer ball, allowing inserting bigons.
- Can you classify/list all *symmetric* such colorings of the 1-skeleton of the soccer ball, that is, the ones that admit at least one nontrivial automorphism?
- Orienting and coloring the 1-skeleton of some platonic solids, allowing inserting bigons. Examples of symmetric colorings. Examples Examples of symmetric realizations of the 1-skeleton of some (or all) platonic solids.
- Can you classify/list all *symmetric* such colorings of the 1-skeleta of some (or all) platonic solids?

Homework 2. Due at 2pm on Friday, September 26, 2025.

(0) (No star means not to turn in.) Given any subset $A \subseteq X \times X$, prove that $\widehat{A}$ is an equivalence relation on X.

(1*) Describe in some way all standard transformations of the 3-dimensional cube, including reflections. What is the dimension of the (standard, euclidean) space where the cube should be lifted to guarantee that each transformation can be performed as a physical rotation within that space? Can you try to find the smallest such dimension?

(2*) Show that the nested union of equivalence relations on a set X,

$$\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \ldots \subseteq X \times X,$$

is again an equivalence relation on X.

(3*) If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq X \times X$ is a nested sequence of arbitrary subsets of $X \times X$, prove that

$$\bigcup_{i=0}^{\infty} \widehat{A_i} = \widehat{\bigcup_{i=0}^{\infty} A_i}.$$

(Here $\widehat{A}$ represents the equivalence relation generated by $A \subseteq X \times X$.) (State clearly what it means for two sets to be equal, then prove the equality of two sets.)

(4*) Let $X_0, X_1, X_2, \ldots$ be a sequence of arbitrary sets, and denote

$$Y_n := \bigcup_{i=0}^{n} X_i, \qquad X := \bigcup_{i=0}^{\infty} X_i = \bigcup_{n=0}^{\infty} Y_n.$$

Let $A_0, A_1, A_2, \ldots \subseteq X \times X$ be a sequence of arbitrary subsets of $X \times X$, not necessarily nested, such that $A_i \subseteq X_i \times X_i$ for each i. Denote $B_n := \bigcup_{i=0}^{n} A_i \subseteq Y_n \times Y_n$. Then we get a nested sequence

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq X \times X.$$

Prove that

$$\widehat{\bigcup_{i=0}^{\infty} A_i} = \widehat{\bigcup_{i=0}^{\infty} B_i} = \bigcup_{i=0}^{\infty} \widehat{B_i}.$$

Here $\widehat{B_i}$ means the completion of B_i $\underline{\text{in } Y_i \times Y_i}$, and the other two completions $\widehat{\ }$ are taken in $X \times X$.

(5*) Explain how the equality in (4) implies that two different definitions of limit for a sequence of sets give the same result (quotient). Then explain how the same can be said about graphs (instead of sets).

For extra fun.

- Fix some set S of colors. If we pick a finite set of starting polygons *randomly* and assign orientations to its edges *randomly* and assign colors to the edges *randomly*, and then run the gluing procedure to the limit, then how often will the result be an *infinite* graph (or infinite 2-complex)?

Week 6. An abstract edge, $\mathbb{E} \subseteq \mathsf{AbsEdges}$. An abstract path P, $\mathbb{P} \subseteq \mathsf{AbsPaths}$. An abstract loop, $\mathbb{L}_2 \subseteq \mathsf{AbsLoops}$. An abstract line, $\mathbb{L}_1 \subseteq \mathsf{AbsLines}$.

Different versions of the gluing construction from different building blocks. Two other ways to obtain the free ground over S: from AbsPaths, from AbsLines.

Creating a universe. Let there be the light! The mixed gluing construction using abstract edges and abstract loops: $\mathbb{E} \subseteq \mathsf{AbsEdges}$ and $\mathbb{L}_2 \subseteq \mathsf{AbsLoops}$. The mixed gluing construction using abstract lines and abstract loops: $\mathbb{L}_1 \subseteq \mathsf{Lines}$ and $\mathbb{L}_2 \subseteq \mathsf{AbsLoops}$. Comparison of the two mixed gluing constructions. The cases $S \cap T = \emptyset$ and $S \cap T \neq \emptyset$. In the case $S \cap T = \emptyset$ the gluing construction using free grounds, $\underline{F}(S) * \underline{F}(T)$ for $\underline{F}(S), \underline{F}(T) \in \mathsf{FreeGrounds}$.

Three ways to construct the free ground $\underline{F}(S)$. The fourth way: from free grounds. Any nested union of trees is a tree. The free product of lines $\underline{\mathbb{Z}}(s_1) * \underline{\mathbb{Z}}(s_2) * \ldots * \underline{\mathbb{Z}}(s_n) = \underline{F}(s_1) * \underline{F}(s_2) * \ldots * \underline{F}(s_n)$.

Week 7. The law of gravity. Amalgamation between two free grounds. The free product of two free grounds. $\underline{F}(S) * \underline{F}(T) \cong \underline{F}(S \sqcup T)$. More generally, for $\mathbb{F} \subseteq \mathsf{FreeGrounds}$. A homogeneous graph (= locally homogeneous graph). Lines are homogeneous. Constructing new homogeneous graphs from lines. Example: identify v_{-1} with v_1 in the line $\underline{\mathbb{Z}}(t)$, then fold. Example: identify v_{-2} with v_1, and v_{305} with v_{100} in $\underline{\mathbb{Z}}(t)$, then fold. (The **PathForms** game is secretly related to this construction. Play the rank 1 level of the game, try find out the relation to the above example.)

A colored graph over S. $\underline{F}(S)$ is homogeneous. The difference between $\mathsf{Paths}(\underline{F}(S))$ and $\mathsf{AbsPaths}(S)$. Constructing new homogeneous graphs from $\underline{F}(S)$ using $R \subseteq \mathsf{Paths}(\underline{F}(S))$. The equivalence relation $\langle R \rangle$ on $\underline{F}(S)$ *generated* by $R \subseteq \mathsf{Paths}(\underline{F}(S))$: the constructive definition.

A graph (equivalence) relation. The flow on a homogeneous graph. A flow-invariant (equivalence) relation on a homogeneous graph $\underline{G}$ over S.

Homework 3. Due at 2pm on Friday, October 10, 2025.

(1*) Given a sequence of sets and inclusions

$$X_0 \overset{\iota_0}{\hookrightarrow} X_1 \overset{\iota_1}{\hookrightarrow} X_2 \overset{\iota_2}{\hookrightarrow} X_3 \overset{\iota_3}{\hookrightarrow} \dots,$$

give the definition of $\lim_{i\to\infty} X_i$ (as a particular quotient). ("Inclusion" here means "an injective function", not necessarily an inclusion of sets.)

(2*) Given a sequence of sets and functions

$$X_0 \overset{\alpha_0}{\to} X_1 \overset{\alpha_1}{\to} X_2 \overset{\alpha_2}{\to} X_3 \overset{\alpha_3}{\to} \dots,$$

describe a natural way to put an equivalence relation on the *disjoint* union $X := \bigsqcup_{i=0}^{\infty} X_i$. Prove that it is indeed an equivalence relation. State a reasonable definition of $\lim_{i\to\infty} X_i$.

(3*) The free ground $\underline{F}(S)$ is a particular colored graph that was defined in class as the limit of a particular gluing process using abstract edges. Prove that any loop (= edge-loop) in $\underline{F}(S)$ is either trivial or not reduced. (This property, plus connectedness, can be taken as a definition of a tree in the category of directed graphs.)

(4*) Prove that the free ground $\underline{F}(S)$ is *transitive* in the sense that
   (a) for any two vertices $u, u' \in V(\underline{F}(S))$ there is an automorphism (= self-isomorphism) $f : \underline{F}(S) \to \underline{F}(S)$ such that $f(u) = u'$, and
   (b) for any two edges $e, e' \in E(\underline{F}(S))$ of the same color there is a color-preserving automorphism $f : \underline{F}(S) \to \underline{F}(S)$ such that $f(e) = e'$.

(Make sure to define f first, show that it is well-defined and also prove that is it indeed an automorphism of a graph onto itself.)

(5*) Recall that the free ground $\underline{F}(S)$ was defined by constructing it starting from a particular vertex, let's call it v_0. Let $\underline{F}$ and $\underline{F}'$ be two disjoint copies of the same ground $\underline{F}(S)$, and let $u \in V(F)$ and $u' \in V(F')$ be arbitrarily chosen vertices, possibly different from v_0 in each copy. Take the disjoint union $\underline{F} \sqcup \underline{F}'$, identify $u \in V(\underline{F})$ with $u' \in V(\underline{F}')$ by putting an appropriate equivalence relation on $\underline{F} \sqcup \underline{F}'$, then keep performing foldings whenever there is a fold. Describe the end result (limit) of this construction. Provide a proof justifying the answer.

For extra fun.

- Fix some set S of colors and a set of lines $L \subseteq \mathsf{Lines}$. Run the gluing construction using (multiple copies of lines in) L. At each step we need to enumerate the vertices that are still not completed. Describe explicitly the enumeration process of vertices using generalized queues that guarantees that each vertex will be completed at some step in the process. Note that each line is infinite, so the queue will have to be supplied with infinitely many new vertices and appropriately reenumerated at each step. Describe how this can be done.

- Describe the queue-enumerating procedure for the gluing construction using free grounds as building blocks that guarantees that each vertex will be completed at some step in the gluing process. (It might be easier too do this using *lines* first, those are free grounds of rank 1. Then generalize to free grounds of arbitrary ranks.)
- For the definition of limit given in (1) and (2) in homework 3 above, state and prove the universal property for this limit.

Week 8. The equivalence relation $\langle\mathcal{R}\rangle$ on $\underline{F}(S)$ *generated* by $\mathcal{R} \subseteq \mathsf{Paths}(\underline{F}(S))$: the abstract definition. (The minimal flow-invariant graph equivalence relation for which endpoints of paths in $\mathcal{R}$ are eqivalent.) (We emphasize that $\langle\mathcal{R}\rangle$ is an equivalence relation, not a subset.) The constructive definition of $\langle\mathcal{R}\rangle$: by gluing the endpoints of paths in $\mathcal{R}$ and then performing foldings. *A subground* of $\underline{F}(S)$ is any equivalence relation $\langle R\rangle$ for some $R \subseteq \mathsf{Paths}(\underline{F}(S))$: the constructive definition. The second definition of a path in $\underline{F}(S)$: a function $p : P \to \underline{F}(S)$. The transfer of a path from one vertex to another, $\mathrm{tr}(p, v')$ (possibly between different homogeneous colored graphs over S.)

The transfer equivalence on $\mathsf{Paths}(\mathcal{G})$ for a homogeneous graph $\mathcal{G}$, the transfer equivalence of paths in different homogeneous graphs over S. The equivalence relation $\langle\langle R\rangle\rangle$ on $\underline{F}(S)$ *transfer-generated* by R (= normally generated by R): the constructive definition. $\langle\langle R\rangle\rangle = \langle \mathrm{Tr}(R)\rangle$. The difference between $\underline{F}(S)/\langle R\rangle$ and $\underline{F}(S)/\langle\langle R\rangle\rangle$ for $R := \{[a,b]\}$. (Here $[a,b]$ is meant to represent *one* concrete path in $\underline{F}(a,b)$.) A transfer-invariant (equivalence) relation $\sim$ on $\underline{F}(S)$.

$\langle\langle R\rangle\rangle$ from constructive definition is transfer-invariant (idea of proof for now). The equivalence relation transfer-generated by R, $\langle\langle R\rangle\rangle$ on $\underline{F}(S)$: the abstract definition. Example: $\underline{F}(S)/\langle$one path with label $aba^{-1}b^{-1}\rangle$ and $\underline{F}(S)/\langle\langle$one path with label $aba^{-1}b^{-1}\rangle\rangle$. A loop-homogeneous graph (= globally homogeneous graph). (A *regular graph* might be another appropriate name for this notion, to indicate the analogy with regular covering spaces in algebraic topology, but this term is probably better reserved to mean *generalized grounds* similar to the state-symmetry graphs we saw at the beginning of the course.)

Week 9. $\underline{F}(S)$ is loop-homogeneous (hw). Constructing new loop-homogeneous graphs from $\underline{F}(S)$. Four definitions of *a ground* $\underline{G}$ over S (the order is interchanged from originally presented):

- (Constructive 1.) The result of transfer-invariant self-gluing of $\underline{F}(S)$, constructive $\lim_{i\to\infty} \underline{F}(S)/\widehat{R_i}$ or abstract $\underline{F}(S)/\langle\langle R\rangle\rangle$, over some set of (concrete) paths $R \subseteq \mathsf{Paths}(\underline{F}(S))$.
- (Constructive 2.) The result of the mixed gluing procedure using both *all* the abstract edges in $\mathsf{AbsEdges}(S)$ and *some* abstract loops $R \subseteq \mathsf{AbsLoops}(S)$.
- (Abstract 1.) A connected loop-homogeneous (colored) graph over S.
- (Abstract 2.) A connected, homogeneous, transitive graph over S.

For any homogeneous graph $\mathcal{G}$ over S there exists a surjective color-preserving graph map $\underline{F}(S) \twoheadrightarrow \mathcal{G}$. (This is part of (3) $\Rightarrow$ (1).) Grounds do not have any preferred elements (unlike groups). Using $\mathcal{R} \subseteq \mathsf{Paths}(\underline{F}(S))$ or $\mathcal{R} \subseteq \mathsf{AbsPaths}(S)$ or $\mathcal{R} \subseteq \mathsf{AbsLoops}(S)$ to define a ground. ~~A group presentation.~~ A ground presentation $\langle S|\mathcal{R}\rangle$. Grounds with one *cyclically reduced* relator $r \in \mathsf{AbsPaths(S)}$, $r \to \underline{F}(S) \twoheadrightarrow \underline{F}(S)/\langle\langle r\rangle\rangle$. Does loop-injectivity hold for one-relator grounds?

Multigrounds over S - another benefit over groups. Grounds are constructive and geomegric precursors to groups. $\mathsf{Grounds}(S)$ and $\mathsf{Grounds}$. ~~The usual definition of a group~~. The *partially defined* product $* : \mathsf{Paths}(\underline{G}) \times \mathsf{Paths}(\underline{G}) \dashrightarrow \mathsf{Paths}(\underline{G})$, $(p, p') \mapsto p * p'$. The relation $\sim_{\underline{G}}$ on $\mathsf{Paths}(\underline{G})$: $p \sim_{\underline{G}} p'$ if after the transfer of the first path to the initial vertex of the second path, their terminal points coincide, that is, $\tau(\mathrm{tr}(p, \iota(p'))) = \tau(p')$. (Equivalently, the induced *flow maps* $\mathrm{flow}_p, \mathrm{flow}_{p'} : V(\underline{G}) \to V(\underline{G})$ coincide.) The induced product $\cdot : \mathsf{Paths}(\underline{G})/\sim_{\underline{G}} \times \mathsf{Paths}(\underline{G})/\sim_{\underline{G}} \to \mathsf{Paths}(\underline{G})/\sim_{\underline{G}}$ (the specific definition and a general homogeneous definition).

$(\mathsf{Paths}(\underline{G})/\sim_{\underline{G}}, \cdot)$ is a group (hw). $\mathrm{gr}(\underline{G}) := \mathsf{Paths}(\underline{G})/\sim_{\underline{G}}$. (Play **PathForms** to see how paths lead to words and vice versa.) The label of a path over S $\leftrightarrow$ a transfer-equivalence class of paths in a homogeneous graph over S. The equivalence relation $\langle R \rangle_{\underline{G}}$ on an arbitrary ground $\underline{G}$ generated by $R \subseteq \mathsf{Paths}(\underline{G})$. *A subground* of an arbitrary ground $\underline{G}$ is the equivalence relation $\langle R \rangle_{\underline{G}}$ on $\underline{G}$ for some $R \subseteq \mathsf{Paths}(\underline{G})$. The subground of an arbitrary ground $\underline{G}$ transfer-generated by R, $\langle\langle R \rangle\rangle_{\underline{G}}$ (= generated by $Tr(R)$, $\langle Tr(R) \rangle_{\underline{G}}$).

Week 10. Analogy between $\mathrm{gr}(\underline{G}) := \mathsf{Paths}(\underline{G})/\sim_{\underline{G}}$ and vectors for a ground $\underline{G}$. Grounds combine features of groups and spaces. (There is a bijection $V(\underline{G}) \to \mathrm{gr}(\underline{G})$, but it is not canonical and looses homogeneity of $\underline{G}$.) Studying graphs with a metric prospective: isometries and quasiisometries. The path metric on $V(\mathcal{G})$ and on $\mathcal{G}$. Word metric vs. path metric.

Two definitions of a quasiisometry. The *correct* (abstract and unrevealing) definition of the Cayley graph $\mathcal{G}(G, S)$ for a given group G and a generating set S.

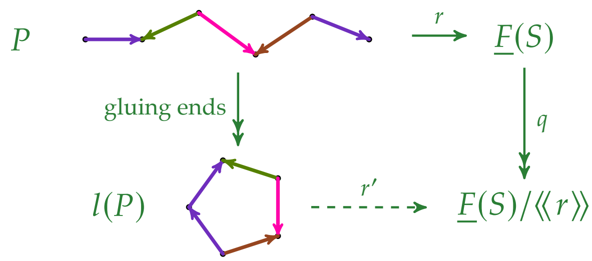
Cayley graphs non-isomorphic *as graphs* (but: quasiisometric and also will be isomorphic *as grounds*). Edges (g, s) in the Cayley graph (and also elements $s \in S$) can be interpreted as transformations $g \mapsto gs$ (or as "flow from g along s"). Projections $\underline{F}(S' \sqcup S'') \twoheadrightarrow \underline{F}(S')^{+S''}$. Rewriting maps $\rho : S \to \mathsf{AbsPaths}(T)$. *The transition* $\mathrm{tr}_{v_0, v_1} : \underline{G} \to \underline{G}$ of a ground $\underline{G}$ induced by transfers of paths. (Refers to being *transitive*, better name than "deck transformation".) Two definitions of a *ground homomorphism* $\underline{\varphi} : \underline{G} \dashrightarrow \underline{H}$ between two grounds:

(1) (Constructive - by *connecting dots* via ρ.) It is a map $\underline{\varphi} : V(\underline{G}) \to V(\underline{H})$ corresponding to (compatible with) some rewriting map $\rho : S \to \mathsf{AbsPaths}(T)$. (*If* such $\underline{\varphi}$ exists, or *if* "ρ rewrites loops to loops", then $\underline{\varphi}$ can be constructed algorithmically - possibly in infinite time - starting from one vertex.)
(2) (Abstract.) It is a transition-equivariant map $\underline{\varphi} : V(\underline{G}) \to V(\underline{H})$. (I.e., a map commuting with the transitions.)

Projections are *strong* homomorphisms of grounds. A strong homomorphism between grounds.

For extra fun.

(1) *Loop injectivity for one-relator grounds.* Consider one *cyclically reduced* path r in a free ground $\underline{F}(S)$, realize it as a map of an abstract path P, $r : P \to \underline{F}(S)$, then turn P into a (cyclically reduced) abstract loop $l(P)$ by gluing the endpoints of P and consider the induced, color-preserving loop (map) $r' : l(P) \to \underline{F}(S)/\langle\langle r \rangle\rangle$ forming the following commutative diagram



where q is the quotient map between grounds.

Can you prove that the loop map r' is injective? A proof of this statement is claimed by one article from 1970, but in a very different language: the statement is for one-relator groups and the proof uses groups, group presentations, and an obscure algebraic argument. I am not convinced that that proof is correct. Can you find a nice, geometric proof of loop injectivity? Or disprove the statement.

(2) Let S be a finite set of colors. Write out the algorithm that determines the (transfer-invariant) equivalence relation $\langle\!\langle R\rangle\!\rangle$ on $\underline{F}(S)$. The transfers of paths in R, the identifications of their endpoints and foldings needs to be performed in such an order that guarantees that in the limit all endpoints of all transfers of all paths in R are identified, and all the foldings are performed (so not folds are left).

(3) Extend the above process to the case when S is infinite and countable. Is it posible to generalize this process to a set S of any cardinality?

(4) Suppose that R is a set of paths in a free ground $\underline{F}(S)$, and R' is obtained by replacing each path $r \in R$ with a *conjugate* of r, meaning a concatenation $w_r r w_r'$, where w_r and w_r' are some paths in $\underline{F}(S)$, w_r' is a transfer of w_r, and the terminal points of w_r and w_r' coincide with $\iota(r)$ and $\tau(r)$, respectively. (Draw the picture.) Prove that $\langle\!\langle R\rangle\!\rangle = \langle\!\langle R'\rangle\!\rangle$ on $\underline{F}(S)$.

(5) Generalize (4) to arbitrary grounds.

(6) Show how to extend the path metric from the vertices of any connected graph $\mathcal{G}$ to the full graph (geometric realization) $\mathcal{G}_{geom}$.

Homework 4. Due at 2pm on Friday, October ~~13~~ 31, 2025. 👻 ☠ 🐈 🧙 🕷 🌙 🐦 🎃 💀 Staple the pages. 💀

(1) Prove the equivalence of two definitions of the equivalence relation $\langle R\rangle$ on $\underline{F}(S)$, constructive and abstract. Then generalize to any loop-homogeneous graph.

(2) Prove that the equivalence relation $\langle R\rangle$ on $\underline{F}(S)$ given by the constructive definition is a graph equivalence relation and is flow-invariant.

(3) Prove that the equivalence relation $\langle\!\langle R\rangle\!\rangle$ on $\underline{F}(S)$ given by the constructive definition is independent of the order of vertex identifications and of the order of foldings, provided that all the necessary identifications and all the necessary foldings have been performed during the process.

(4) Prove that the equivalence relation $\langle\!\langle R\rangle\!\rangle$ on $\underline{F}(S)$ given by the constructive definition is a graph equivalence relation, is flow-invariant and transfer-invariant.

(5) Prove the equivalence of two definitions of the equivalence relation $\langle\!\langle R\rangle\!\rangle$, constructive and abstract. Then generalize to any loop-homogeneous graph.

(6*) The free ground $\underline{F}(S)$ is a particular colored graph that was defined (constructed from abstract edges) in class before. Prove that $\underline{F}(S)$ is loop-homogeneous (= globally homogeneous). (By definition, being loop-homogeneous in particular involves being homogeneous.) (Hint added later: there are many ways to do this, here is one possible way. (a) First use the recursive construction of $\underline{F}(S)$ to prove the following: any loop in $\underline{F}(S)$ is either trivial (= has one vertex and no edges) or not reduced. That is, any nontrivial loop must have backtracking. That is, any reduced loop must be trivial. (Any one of these properties can be taken as a definition of a tree.) (b) Then check that *reduction* (= removing backtracking) does not change the end points of a path, and use this to prove that for a homogeneous graph loop-homogeneity can be *equivalently* defined using only *reduced* loops. Combining (a) and (b), reduce the problem to proving that in $\underline{F}(S)$, transfers of *trivial* loops are again loops. This last statement is true.)

(7) Consider any set of paths $R \subseteq \mathsf{Paths}(\underline{F}(S))$. Using (6), prove that $\underline{F}(S)/\langle\langle R \rangle\rangle$ is loop-homogeneous.

(8*) Let $\underline{G}$ be any loop-homogeneous graph over S. Prove that the relation $\sim_{\underline{G}}$ on $\mathsf{Paths}(\underline{G})$ defined in class is indeed an equivalence relation. (As usual, it helps to draw pictures.)

(9*) Let $\underline{G}$ be any loop-homogeneous graph over S. Prove that the binary operation $\cdot$ on the quotient $\mathsf{Paths}(\underline{G})/{\sim_{\underline{G}}}$ defined in class is well-defined. (It helps to draw pictures.)

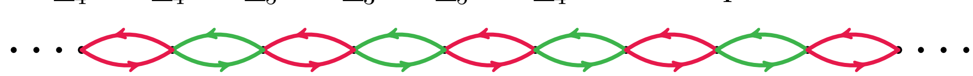
(10*) Let $\underline{G}$ be any loop-homogeneous graph over S. (Here we do not need to assume that $\underline{G}$ is connected, so in particular, any multiground *over S* will do.) Prove that the quotient $G := \mathrm{gr}(\underline{G}) := \mathsf{Paths}(\underline{G})/{\sim_{\underline{G}}}$ is a group with respect to the operation $\cdot$. (It helps to draw pictures.) In particular, using (6), deduce that $F(S) := \mathsf{Paths}(\underline{F}(S))/{\sim_{\underline{F}(S)}}$ is a group.

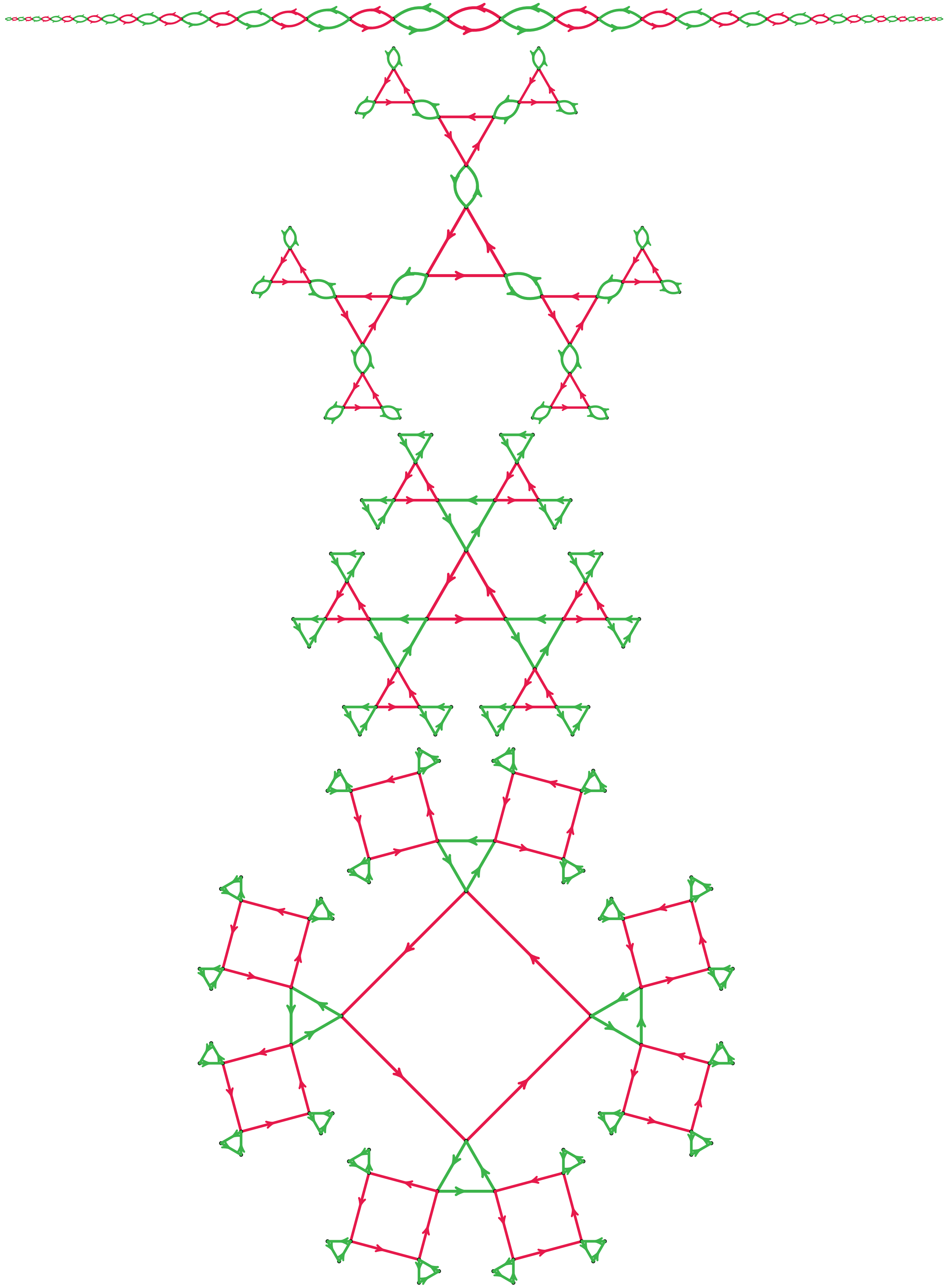
($F(S)$ is called *the free group* with basis S, and usually it is defined without using grounds. The proof of associativity for $F(S)$ usually uses obscure, unrevealing algebraic arguments. Here we give a constructive and geometric definition of the free group, and the proof of associativity should be clear and explicit from the geometry of the free ground $\underline{F}(S)$. Also, it is a good, and easy, exercise to check that the usual definition of $F(S)$ (find it) and the above definition of $F(S)$ give isomorphic groups; again, the geometry of the free ground $\underline{F}(S)$ helps to construct such an isomorphism.)

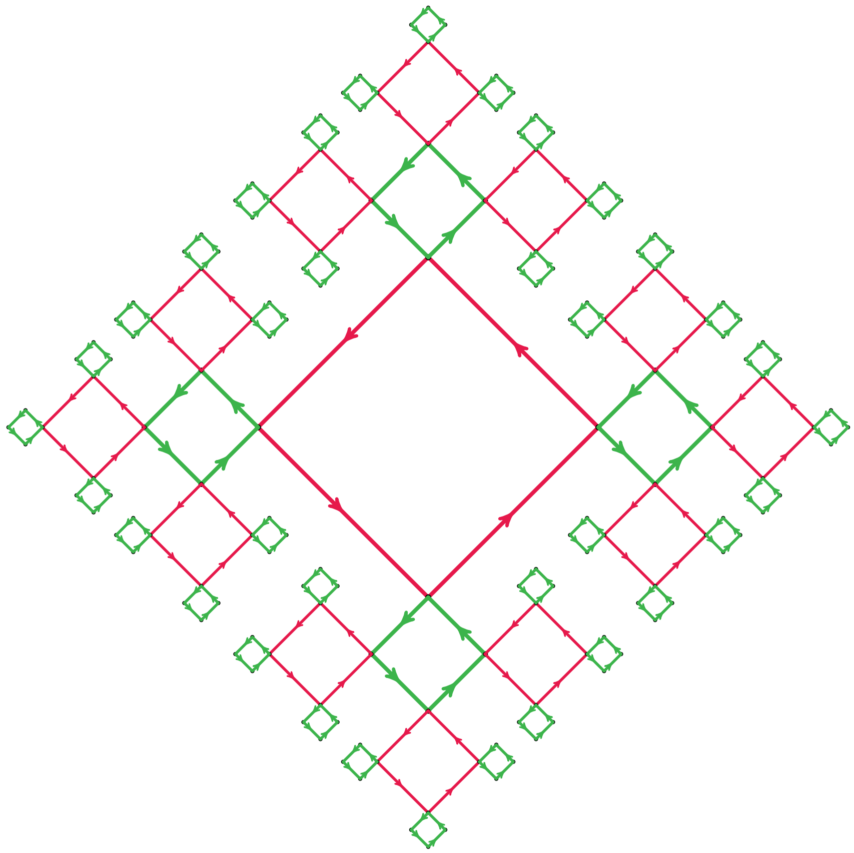
Week 11. Discussing the two definitions of ground homomorphisms (by connecting dots and by transition-equivariance). The formal definition of an arbitrary transition $\mathrm{tr}_{v_0,v_1} : \underline{G} \to \underline{G}$ of an arbitrary ground $\underline{G}$.

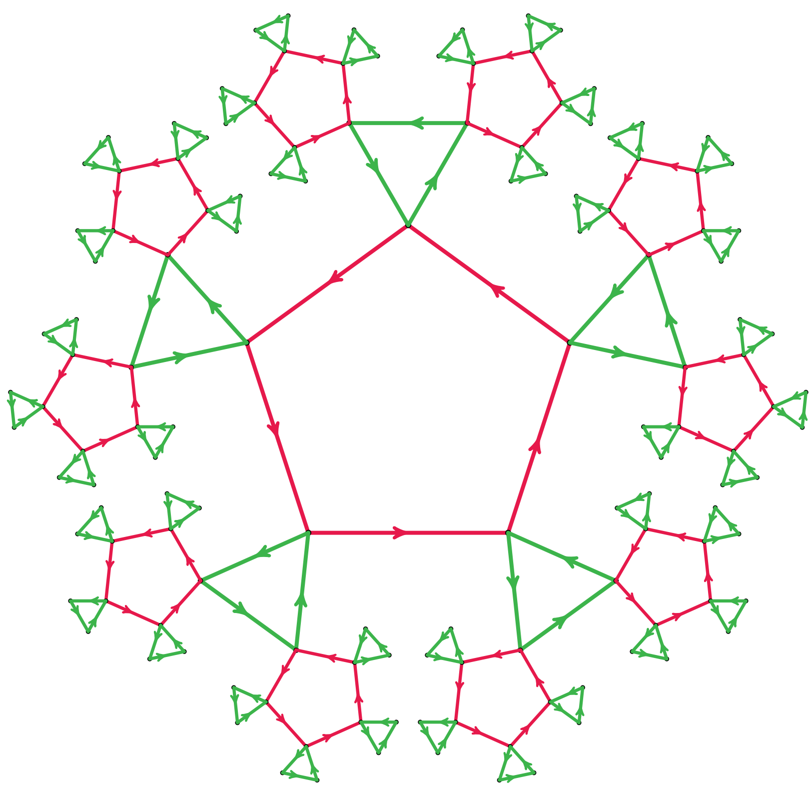
Analogy between ground homomorphisms and linear maps between vector spaces. The rewriting map ρ is analogous to the map of basis vectors (or rather, of spanning vectors). Ground homomorphisms and ground isomorphisms. To define $\underline{F}(S') * \underline{F}(S'')$ constructively: make the colors disjoint first, as done in the disjoint union. How to define the free product of grounds?

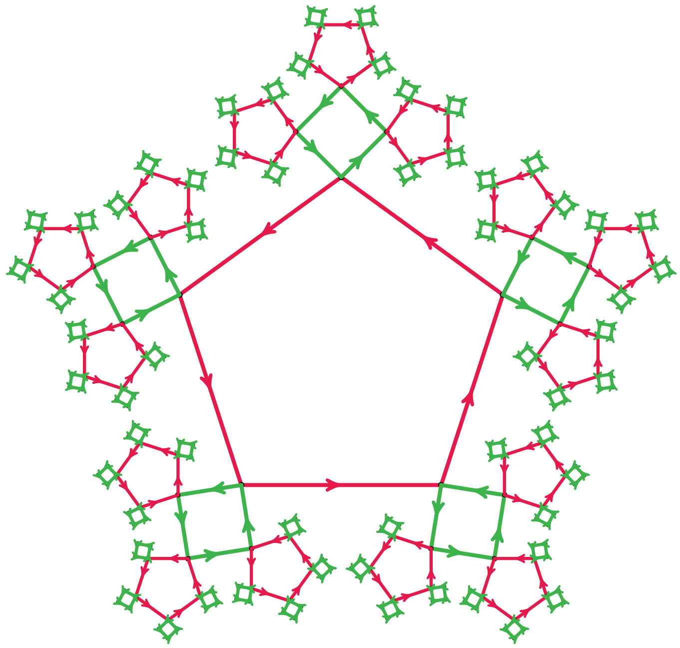
A *finitely generated* subground $\sim$ of $\underline{G}$. *The free product* of arbitrary grounds $\underline{G} * \underline{H}$: the constructive-geometric definition. (Here “free” should mean “free of additional loops”.) (The way the corresponding notion for groups - the free product of groups - is defined is abstract and unrevealing.) The free product of grounds *is* a ground. Tree-like structures. $\underline{\mathbb{Z}}_n[s] := \underline{\mathbb{Z}}[s]/\langle\langle \text{one path labeled } s^n \rangle\rangle$. Construct $\underline{\mathbb{Z}}_2[s] * \underline{\mathbb{Z}}_2[t]$, $\underline{\mathbb{Z}}_3[s] * \underline{\mathbb{Z}}_2[t]$, $\underline{\mathbb{Z}}_3[s] * \underline{\mathbb{Z}}_3[t]$, $\underline{\mathbb{Z}}_4[s] * \underline{\mathbb{Z}}_3[t]$, $\underline{\mathbb{Z}}_4[s] * \underline{\mathbb{Z}}_4[t]$, $\underline{\mathbb{Z}}_5[s] * \underline{\mathbb{Z}}_3[t]$, $\underline{\mathbb{Z}}_5[s] * \underline{\mathbb{Z}}_4[t]$, see the pictures below.











Using grounds to give a better (constructive, geometric) definition of *the free product of groups*:

Cayley graphs $\mathcal{G} := \mathcal{G}(G,S)$, $\mathcal{H} := \mathcal{G}(H,T)$ are grounds. ⟵ Groups G, H.

(this definition is geometric and constructive) ↓ ? ↓ (the traditional definition is algebraic and abstract)

Free product of grounds $\mathcal{G} * \mathcal{H}$ is a ground. ⟶ We define the free product of groups $G * H$ to be the group $\mathrm{gr}(\mathcal{G} * \mathcal{H})$ associated with the ground $\mathcal{G} * \mathcal{H}$.

This construction is independent (up to a ground isomorphism) of the choice of generating sets S and T.

Week 12. The normal subground $\langle\langle \mathcal{R} \rangle\rangle_{\underline{G}}$ can be equivalently defined using abstract paths $\mathcal{R} \subseteq \mathsf{AbsPaths}(S)$ in place of concrete paths. Using grounds to give a better (constructive, geometric) definition of groups associated with group presentations: given S and R, define

$$\mathcal{R} := \{\text{the abstract paths over } S \text{ corresponding to words in } R\},$$

$$\mathrm{gr}\langle S|R\rangle := \mathrm{gr}(\underline{\mathrm{gr}}\langle S|\mathcal{R}\rangle) := \mathrm{gr}(\underline{F}(S)/\langle\langle \mathcal{R} \rangle\rangle).$$

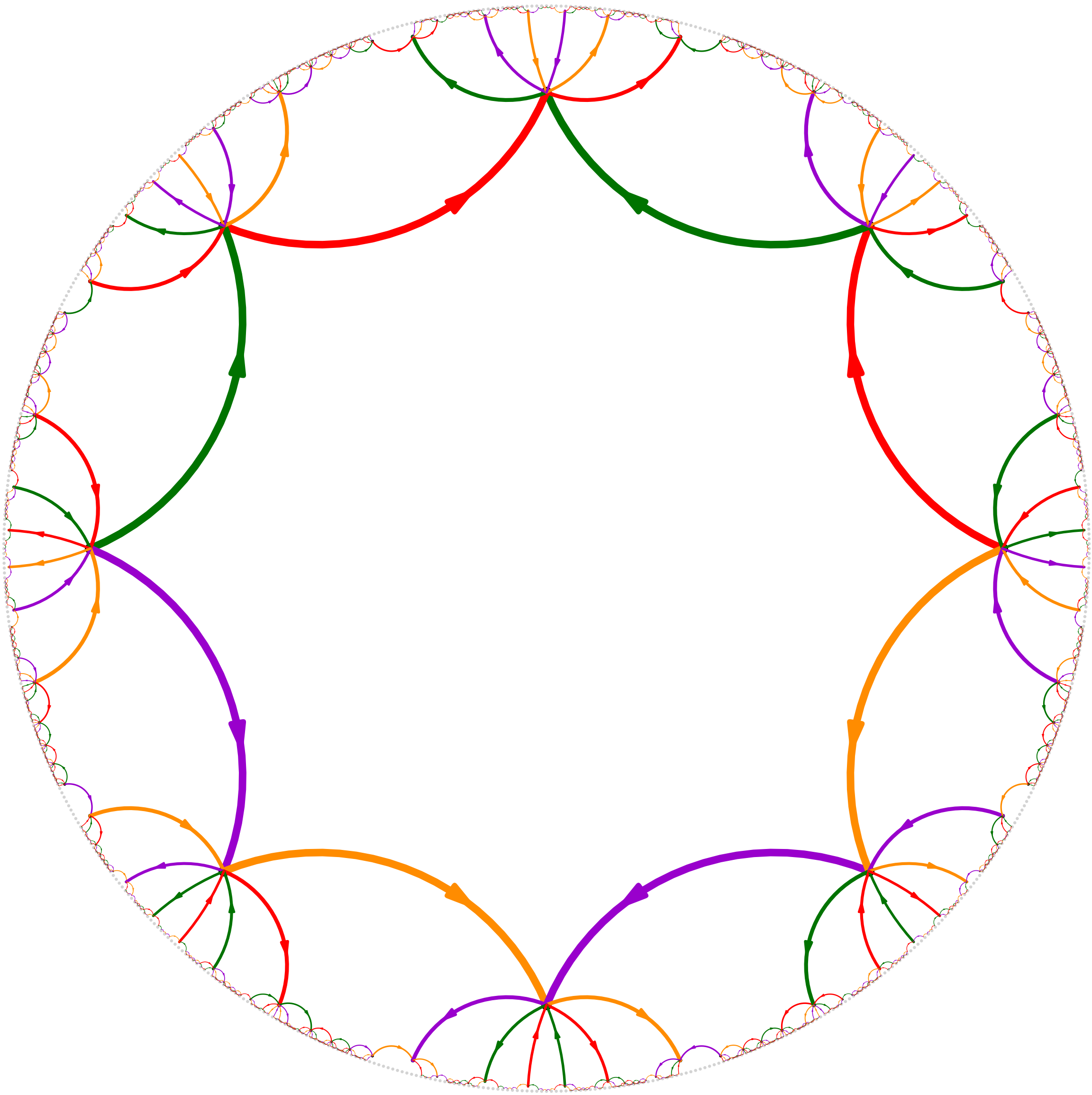
Ground presentation $\langle S|\mathcal{R}\rangle$. ⟵ Group presentation $\langle S|R\rangle$.

(this definition is geometric and constructive) ↓ ? ↓ (the traditional definition is algebraic and abstract)

Ground $\underline{\mathrm{gr}}\langle S|\mathcal{R}\rangle := \underline{F}(S)/\langle\langle \mathcal{R} \rangle\rangle$ can be *constructed* from the ground presentation $\langle S|\mathcal{R}\rangle$ (in two ways). ⟶ We define the group $\mathrm{gr}\langle S|R\rangle$ associated with a presentation $\langle S|R\rangle$ to be $\mathrm{gr}(\underline{\mathrm{gr}}\langle S|\mathcal{R}\rangle)$.

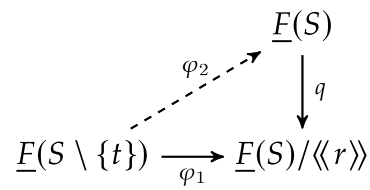
("gr" stands for "group", "$\underline{\mathrm{gr}}$" stands for "ground".) Embeddings $\underline{F}(S') \hookrightarrow \underline{F}(S)$ for $S' \subseteq S$.

The general free product of a family of grounds $*_{i \in I} \underline{G}_i$. *The surface ground* of genus g: color the sides of one $4g$-gon in pairs and glue multiple copies of the $4g$-gon, upstairs. Below is the surface ground of genus 2, i.e, the ground constructed from copies of one octagon (using constructive definition 2 of grounds). The first 2000 octagons of the gluing process are shown. What is left from this surface ground when all the edges of one color are removed?



(Escher's picture below is another illustration to this type of behavior.)

Magnus' *Freiheitssatz* in the language of grounds. (Also see (12) and (13) in hw5 below.)



(There is a related story from Max Dehn, Wilhelm Magnus, Paul Schupp.)

Two constructive definitions of *a one-relator ground* (following the two constructive definitions of a ground):

- (Constructive 1.) The result of the mixed gluing procedure using (multiple copies of) both *all* the abstract edges in $\mathsf{AbsEdges}(S)$ and *one* abstract loop $r \in \mathsf{AbsLoops}(S)$.
- (Constructive 2.) The result of transfer-invariant self-gluing of $\underline{F}(S)$ using one path $r \in \mathsf{Paths}(\underline{F}(S))$ (or an abstract path $r \in \mathsf{AbsPaths}(S)$), that is, $\underline{F}(S)/\langle\langle r \rangle\rangle$.

(The two abstract definitions of a ground cannot be used for this definition.) Another example of one-relator grounds: check the *Freiheitssatz* for the ground presentation $\langle a, b |$one path labeled $[a, b]\rangle$.

Week 13. Each projection $\underline{F}(S' \sqcup S'') \twoheadrightarrow \underline{F}(S')^{+S''}$ induces an equivalence relation $\sim$ on $\underline{F}(S' \sqcup S'')$. (Also see hw.) The projection $\underline{F}(S' \sqcup S'') \to \underline{F}(S')^{+S''}$ and its modification $\underline{F}(S' \sqcup S'') \dashrightarrow \underline{F}(S')$. Two definitions of *a subground* of an arbitrary ground $\underline{G}$:

(1) (Constructive.) It is the equivalence relation $\langle R \rangle_{\underline{G}}$ on $\underline{G}$ for some $R \subseteq \mathsf{Paths}(\underline{G})$.
(2) (Abstract.) It is any graph equivalence relation $\sim$ on $\underline{G}$ that is flow-invariant.

Two definitions of *a transfer-invariant subground* (*=a normal subground*) of an arbitrary ground $\underline{G}$:

(1) (Constructive.) It is the equivalence relation $\langle\langle R \rangle\rangle_{\underline{G}}$ on $\underline{G}$ for some $R \subseteq \mathsf{Paths}(\underline{G})$.
(2) (Abstract.) It is any graph equivalence relation $\sim$ on $\underline{G}$ that is flow-invariant and transfer-invariant.

Two constructive definitions of *the ground associated with a ground presentation* $\langle S|R\rangle$, denoted $\underline{\mathrm{gr}}\langle S|R\rangle$ (parallel to the two constructive definitions of a ground):

- (Constructive 1.) It is the result of the recursive gluing procedure using (multiple copies of) the abstract edges corresponding to the colors in S and the abstract loops corresponding to the paths in R as building blocks.

- (Constructive 2.) It is the result of gluing the free ground $\underline{F}(S)$ to itself according to the paths in R and all of their transfers in $\underline{F}(S)$, and folding all the resulting folds, recursively.

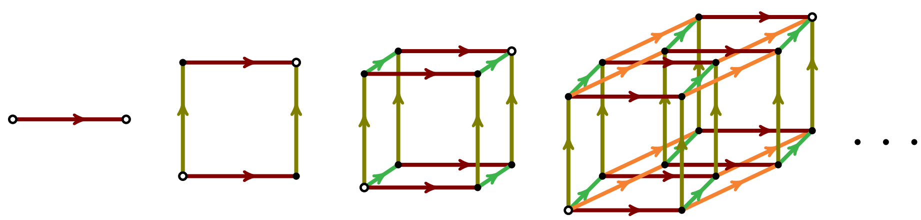
Another example of a one relator ground: check the *Freiheitssatz* for the ground $\underline{\mathrm{gr}}\langle a,b,c|\text{path } [a,b]\rangle$. Construct it using both definitions. Is it a coincidence that it is tree-like? $\underline{\mathrm{gr}}\langle a,b,c|\text{path } [a,b]\rangle \cong \underline{\mathrm{gr}}\langle a,b|\text{path } [a,b]\rangle * \underline{\mathrm{gr}}\langle c|\emptyset\rangle \cong (\underline{\mathbb{Z}}[a]\underline{\times}\underline{\mathbb{Z}}[b]) * \underline{\mathbb{Z}}[c]$.

Describe the ground $\underline{\mathrm{gr}}\langle a,b,c,d|\text{path } [a,b], \text{path } [c,d]\rangle$. Is it tree-like? Why? Can the ground $\underline{\mathrm{gr}}\langle S' \sqcup S''|R\rangle$ split as a free product of grounds? When two grounds are given by presentations: thre constructive definitions of the free product of grounds (parallel to the two constructive definitions of grounds: one from "gluing multiple building blocks", the other from "gluing a free ground"). *A strong ground homomorphism* (= *an edge homomorphism*). Find *a transfer-generating set* (= normally generating set) for the kernel of the projection $q : \underline{F}(a,b) \to \underline{F}(a)^{+b}$, i.e., $R \subseteq \mathsf{Paths}(\underline{F}(a,b))$ such that $\operatorname{Ker} q = \langle\langle R\rangle\rangle$ (hw).

For extra fun.

(1) For any set S of colors, write a formal definition for *the product* of infinite cyclic grounds $\underline{\prod}_{s\in S}\underline{\mathbb{Z}}[s]$ using the picture below as inspiration. Here S can be of any cardinality. The result should be a colored graph again: describe the its set of vertices, the set of edges, and how the edges are attached to the vertices. First check that $\underline{\prod}_{s\in S}\underline{\mathbb{Z}}[s]$ is a loop-homogeneous graph over S.

Is the product $\underline{\prod}_{s\in S}\underline{\mathbb{Z}}[s]$ connected? That is, is this product a ground or a multi-ground?



(2) Generalizing (1), give a formal definition of the product $\underline{\prod}_{i\in I}\underline{G}_i$ of an arbitrary indexed family of grounds $(\underline{G}_i \mid i \in I)$ over the sets of colors S_i.

(3) What should be the right definition of the direct sum $\underline{\oplus}_{s\in S}\underline{\mathbb{Z}}[s]$? And more generally, what should be the right definition of the direct sum $\underline{\oplus}_{i\in I}\underline{G}_i$ of an arbitrary family of grounds?

(4) In the case when $S = \{a,b\}$, consider the product $\underline{\prod}_{s\in S}\underline{\mathbb{Z}}[s] = \underline{\mathbb{Z}}[a]\underline{\times}\underline{\mathbb{Z}}[b]$ as in (1). There is a color-preserving graph map $\varphi : \underline{F}(a,b) \twoheadrightarrow \underline{\mathbb{Z}}[a]\underline{\times}\underline{\mathbb{Z}}[b]$. Find a transfer-generating set of paths R for the kernel of φ. Which set R is the smallest one?

(5) Suppose $\underline{G}$ is any ground over a *finite* set of colors S, and let $\sim$ be a subground of $\underline{G}$. (This means that $\sim = \langle R\rangle$ for some set $R \subseteq \mathsf{Paths}(\underline{G})$, where R can be either finite or infinite). Suppose that $\sim$ is of *finite index* in $\underline{G}$, meaning that there are only finitely many $\sim$-equivalence classes of vertices in $V(\underline{G})$, or equivalently, that the quotient $V(\underline{G})/\sim$ is finite. Prove that the subground $\sim$ is *finitely generated* (meaning that there exists a *finite* set $R' \subseteq \mathsf{Paths}(\underline{G})$ such that $\sim = \langle R'\rangle$). (This is a geometric analog of the statement that a finite-index subgroup of a finitely generated group is finitely generated. Give a *geometric* proof of this fact, not an algebraic one. That is, use the structure of the ground $\underline{G}$.)

(6) Under the assumptions of (5) above, let H be any $\sim$-equivalence class in $V(\underline{G})$, with the metric restricted from $\underline{G}$. Prove that the inclusion map $H \hookrightarrow \underline{G}$ is a quasiisometry.

(7) Do there exist subgrounds of fintely generated grounds that are *not* finitely generated? (Hint: Construct a surjective ground homomorphism $\underline{F}(a,b) \twoheadrightarrow \underline{F}(a)^{+b}$. The kernel of this homomorphism is a subground of $\underline{F}(a,b)$. Prove that this subground is not finitely generated.)

(8) Let G be the infinite cyclic group generated by $t \in G$, that is,

$$G := gr\langle t\rangle := \{\ldots, t^{-2}, t^{-1}, 1, t, t^2, \ldots\}.$$

   (a) Show that the sets $S_1 := \{t, t^2\} \subseteq G$ and $S_2 := \{t, t^3\}$ are generating sets of G of the same cardinality.
   (b) Show that the corresponding Cayley graphs $\mathcal{G}(G, S_1)$ and $\mathcal{G}(G, S_2)$ are *not* isomorphic as graphs.
   (c) Show that $\mathcal{G}(G, S_1)$ and $\mathcal{G}(G, S_2)$ are quasiisometric as metric spaces with the path metric.
   (d) Show that, after interpreted as grounds, $\mathcal{G}(G, S_1)$ and $\mathcal{G}(G, S_2)$ are isomorphic as grounds. (*An isomorphism of grounds* is a ground homomorphism that is bijective (on vertices). Equivalently, an isomorphism of grounds is a ground homomorphism for which there exists an inverse ground homomorphism.)

(9) Prove the equivalence of the four definitions of a ground (up to isomorphism of colored graphs).

(10) Prove the equivalence of the two definitions of a ground homomorphism.

(11) Play the **PathForms** game for rank 1 or rank 2 or rank 3, and generate several paths in the free ground $\underline{F}(a,b)$ (press the "generate" button). All these paths start from one vertex. Let R be that particular set of paths in the free ground. Describe the quotient graph by the subground generated by R, $\underline{F}(a,b)/\langle R\rangle$. That is, the result of gluing the endpoints of each path and then folding. Is this quotient a ground?

(10) Let S and T be *finite* sets (of colors) and $\underline{G}$ and $\underline{H}$ be grounds over S and T, respectively, with the corresponding path metrics $d_{\underline{G}}$ and $d_{\underline{H}}$. If there is an isomorphism of grounds $f : \underline{G} \dashrightarrow \underline{H}$ (not necessarily an isomorphism of graphs), prove that the same map $f : (V(\underline{G}), d_{\underline{G}}) \longrightarrow (V(\underline{H}), d_{\underline{H}})$ is a quasiisometry.

(11) Give a constructive-geometric proof of the *Freiheitssatz*, that is, one using grounds and their geometry. (Max Dehn intended the proof to be geometric. Be the first; no one has ever done this!) (Note: we are not looking for a restatement of Magnus' algebraic proof in geometric/topological terms - those do exist. Rather, find a new, conceptual, geometric, "ground proof".)

(12) See the pictures above for the free products of grounds $\underline{\mathbb{Z}}_m[s] * \underline{\mathbb{Z}}_n[t]$ for various m and n. Put the path metric on each ground. What are these grounds quasiisometric to?

(13) Consider the surface ground of genus 2 (illustrated above):

$$\underline{G} := \underline{\mathrm{gr}}\langle a_1, b_1, a_2, b_2 | r_2\rangle = \underline{F}(a_1, b_1, a_2, b_2)/\langle\langle r_2\rangle\rangle,$$

where r_2 is one path in $\underline{F}(a_1, b_1, a_2, b_2)$ with label $[a_1, b_1][a_2, b_2] = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$. Observe that $\underline{F}(a_1, b_1, a_2, b_2)$ splits as the free product of free grounds: $\underline{F}(a_1, b_1, a_2, b_2) \cong \underline{F}(a_1, b_1, a_2) * \underline{\mathbb{Z}[b_2]}$, so we can write

$$\underline{G} = \left(\underline{F}(a_1, b_1, a_2) * \underline{\mathbb{Z}}[b_2]\right)/\langle\langle r_2\rangle\rangle.$$

Prove that $\underline{G}$ is *nontrivial*, that is, has more than one vertex. (This statement is a special case of what is called *the Kervaire-Laudenbach conjecure (KLC), restated in the language of grounds, of course.* It is known to be true *in this case*. One way to see this: in this case KLC follows from the *Freiheitssatz*, which is known to be true. It is better to prove this nontriviality geometrically: give a geometric argument that the surface ground of genus 2 constructed and illustrated above is nontrivial.)

(14) Generalize the above statement as follows. Pick any integers m_1, n_1, m_2, and let

$$\underline{G}_{m_1,n_1,m_2} := \big((\underline{\mathbb{Z}}_{m_1}[a_1] * \underline{\mathbb{Z}}_{n_1}[b_1] * \underline{\mathbb{Z}}_{m_2}[a_2]) * \underline{\mathbb{Z}}[b_2]\big)/\langle\!\langle r_2 \rangle\!\rangle.$$

Prove that the ground $\underline{G}_{m_1,n_1,m_2}$ is nontrivial. (This statement is also a special case of the Kervaire-Laudenbach conjecure. In general, KLC is unknown for grounds (or groups) *with torsion*, like the ground $\underline{\mathbb{Z}}_{m_1}[a_1] * \underline{\mathbb{Z}}_{n_1}[b_1] * \underline{\mathbb{Z}}_{m_2}[a_2]$. But KLC can be proved in this case.)

Homework 5. Due at 2pm on Friday, November 21, 2025. Staple the pages.

(1*) Suppose that $\sim$ is a color-preserving graph equivalence relation on $\underline{F}(S)$ (*color-preserving* means that any two equivalent edges have the same color), and also that $\sim$ is flow-invariant. Prove that the quotient graph $\underline{F}(S)/\sim$ can be colored so that it becomes a homogeneous graph over S, and the quotient map $q : \underline{F}(S) \twoheadrightarrow \underline{F}(S)/\sim$ is color-preserving.

(2*) Under the assumptions of (1), prove that there exists $R \subseteq \mathsf{Paths}(\underline{F}(S))$ such that $\sim = \langle R \rangle$. ($\langle R \rangle$ is the subground of $\underline{F}(S)$ generated by R.)

(3) Under the assumptions of (1), prove that the (colored) quotient graph $\underline{F}(S)/\sim$ is loop-homogeneous if and only if there exists $R \subseteq \mathsf{Paths}(\underline{F}(S))$ such that $\sim = \langle\!\langle R \rangle\!\rangle$. ($\langle\!\langle R \rangle\!\rangle$ is the subground of $\underline{F}(S)$ transfer-generated by R.)

(4) Generalize (1), (2), (3) by replacing the free ground with an arbitrary ground $\underline{G}$ over S.

(5) The projection $\underline{F}(a,b) \twoheadrightarrow \underline{F}(a)^{+b}$ determines an equivalence relation $\sim$ on $\underline{F}(a,b)$. Give an explicit definition of a graph equivalent relation $\sim$ on $\underline{F}(a,b)$ such that the quotient map $\underline{F}(a,b) \twoheadrightarrow \underline{F}(a,b)/\sim$ becomes the same (up to an isomorphism of colored graphs) as the projection $\underline{F}(a,b) \twoheadrightarrow \underline{F}(a)^{+b}$. (Added latter: Then using (2) we can deduce that $\sim$ is a subground of $\underline{F}(a,b)$.)

(6*) Is the subground in (5) transfer-invariant? Justify the answer.

(7*) For the subground in (5), does there exist $R \subset \mathsf{Paths}(\underline{F}(a,b))$ such that $\sim = \langle\!\langle R \rangle\!\rangle$? Justify the answer.

(8*) Let G be any group, $S \subseteq G$ be any generating set, and $\mathcal{G}(G,S)$ be the corresponding Cayley graph. To each edge $(g,s) \in G \times S$ assign the label s viewed as a color. Prove that $\mathcal{G}(G,S)$ is a ground over S. (Use any of the four definitions of a ground.)

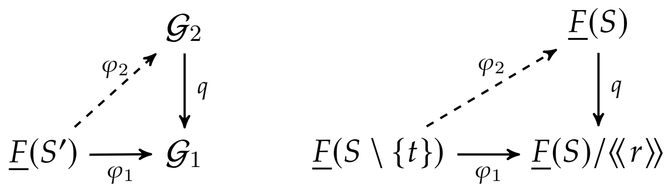
(9) As in (8), start with a group G and a generating set S, consider the corresponding Cayley graph $\mathcal{G} := \mathcal{G}(G,S)$, interpret it as a ground over S, and define the corresponding group $\mathrm{gr}(\mathcal{G}) := \mathsf{Paths}(\mathcal{G})/\sim_{\mathcal{G}}$ as before. Prove that $\mathrm{gr}(\mathcal{G})$ is isomorphic to G, meaning that there is a bijective group homomorphism $\mathrm{gr}(\mathcal{G}) \to G$.

(10) *An open-ended question.* Conversely, start with any ground $\underline{G}$ over S and produce the group $G := \mathrm{gr}(\underline{G}) := \mathsf{Paths}(\underline{G})/\sim_{\underline{G}}$ as before. For this group, let $\mathcal{G}$ be its Cayley graph. How is $\mathcal{G}$ related to the original ground $\underline{G}$? The answer depends on what generating set in G we use to define the Cayley graph $\mathcal{G}$.

(11) *Another open-ended question.* What should be the right definition of *a finitely generated subground* of a ground $\underline{G}$?

(12) Let $\mathcal{G}$ be any homogeneous graph over a set of colors S, and $S' \subseteq S$. Prove that for any vertices $u \in V(\underline{F}(S))$ and $v \in V(\mathcal{G})$ there exists a unique color-preserving map of graphs $\varphi : \underline{F}(S') \to \mathcal{G}$ such that $\varphi(u) = v$.

(13) Let $q : \mathcal{G}_2 \twoheadrightarrow \mathcal{G}_1$ be any surjective color-preserving map between homogeneous graphs over S, and $S' \subseteq S$. Prove that for any color-preserving map $\varphi_1 : \underline{F}(S') \to \mathcal{G}_1$ there exists *a lift*, that is, a map of graphs $\varphi_2 : \underline{F}(S') \to \mathcal{G}_2$ such that $\varphi_1 = q \circ \varphi_2$.



In the special case of the diagram on the right, prove that φ_2 is injective for all $t \in S$. (The injectivity of φ_1 is exactly the statement of the *Freiheitssatz*, assuming that r is cyclically reduced and the color t is present in r.)

Possible topics for projects. Suggestions for other possible topics for project are certainly welcome.

(1) Soccer balls or other interesting symmetric objects with interesting state-symmetry graphs. Examples of symmetric colorings of the 1-skeleton of the soccer ball. Examples of symmetric realizations of the 1-skeleton of the soccer ball, allowing inserting bigons.

(2) Classifying/listing all *symmetric* colorings of the 1-skeleton of the soccer ball, that is, the ones that admit at least one nontrivial automorphism.

(3) Orienting and coloring the 1-skeleton of some platonic solids, allowing inserting bigons. Examples of symmetric colorings. Examples of symmetric realizations of the 1-skeleton of some (or all) platonic solids.

(4) Classifying/listing all *symmetric* colorings of the 1-skeleton of some (or all) platonic solids.

(5) Formal definitions of direct limits (= colimits). Direct limits of sets, of graphs, of other objects. The universal property of the direct limit.

(6) Fix some set S of colors. Investigate possible ways to pick a finite set of starting polygons (or abstract loops) *randomly*. That is, define reasonable notions of randomness (probability measures on the family of finite sets of colored and directed abstract loops). Similarly, investigate various ways to assign orientations to its edges randomly and to assign colors to the edges randomly. With those choices, try to guess how often the result of gluing will be an *infinite* graph (or 2-complex).

(7) Small cancellation conditions for graphs (or 2-complexes) obtained by gluing. Their geometry, properties. Classically, these conditions are defined for groups, explain them in terms of grounds (or the 2-complexes obtained by our gluing procedure, upstairs). (For example, my article *The topology and geometry of units and zero-divisors: origami* uses small-cancellation-type geometric properties to design methods to look for nontrivial units and zero-divisors in group rings.)

(8) Algorithmic problems and properties, for grounds and for groups.

(9) If you like, you might choose a topic of interest from my *Links and comments on things related to geometric group theory*.

(10) Classifying metric spaces, graphs, grounds, up to quasiisometry. Examples of graphs that are quasiisometric. Examples of graphs that are not quasiisometric.
(11) Write a formal definition what *a generalized algorithm* is, for example, a countable-time algorithm that computes a particular graph after a countable infinite number of steps. Define formally what the result of computation (=limit) is. (As with any sequence, the limit does not have to always exist. And as with any recursive process, it does not have to produce any result in general. The goal is to define formally, what we mean by "the result" of an infinite countable algorithm.)
(12) Further generalize the above notion of an algorithm: define a process that can involve the number of steps *of any given cardinality*. *Ordinals* and *transfinite induction* are appropriate notions to use here.
(13) Categories. Define *a category of grounds*, with the morphisms defined using rewriting functions as in class. Prove that it is indeed a category, and describe how this category is related to the category of groups.
(14) Look up *Schreier graphs*. Historically, they have been used to study subgroups of free groups. Later a better language of covering spaces was introduced by algebraic topology. How are Schreier graphs related to the quotients of free grounds discussed in this course?
(15) The notion of *folding* was first introduced by John Stallings. Investigate what he used it for and what folding can further be used for.
(16) Read selected topics from the book by A. Yu. Ol'shanskii *Geometry of defining relations in groups* and present all the results in terms of grounds, and in a constructive-geometric way.
(17) ...

Tentative schedule for the projects:
Group 1: Wednesday, December 3, 2025.
Group 2: Friday, December 5, 2025.
Group 3: Monday, December 8, 2025.
Give the 50-minute presentation in class and hand in the report for the project at the same time. Please practice the presentation in advance to make sure it fits nicely in about 50 minutes. If you use any kind of technology, make sure in advance that it actually works. In the report, write the names of the students who *actually participated* in preparing the project. If you like, you can also hand out copies of the report to other students. All participants in each group will receive the same grade *provided* they contributed to the project. If someone did not contribute, please let me know.

For extra fun.

(1) Take a look at the pictures of various free products $\underline{\mathbb{Z}}_m[s] * \underline{\mathbb{Z}}_n[t]$ above. What is the reason the pieces have to be recursively scaled down in order to fit the free product into the plane? Investigate how this question is related to the question of *volume growth*: define a distance function (path metric) on the free product of grounds and the standard distance in the plane. Define some notion of *volume* on the graph and compare the volume of the ball of radius r in the graph and the volume of the ball of radius r in the plane. What happens when r is large?
(2) For any two transfer-invariant subgrounds (= normal subsgrounds) $M \triangleleft \underline{G}$ and $N \triangleleft \underline{G}$, $M \cdot N = \langle\langle M, N \rangle\rangle_{\underline{G}} = N \cdot M$.

Weeks 14 and 15. (Two classes + three projects.) Plan: kernels, images, free products with amalgamation, HNN-extensions, all stated for grounds rather than for groups. Three possible (nonequivalent) definitions of *a strong ground homomorphism* $\underline{\varphi} : \underline{G} \to \underline{H}$:

(1) $\underline{G}$ is a ground over some S, $\underline{H}$ is a ground over some T, and $\underline{\varphi}$ is a ground homomorphism whose rewriting function $\rho : S \to \mathsf{AbsPaths}(T)$ sends each color to a one-edge abstract path, that is, to an abstract edge.

(2) $\underline{G}$ is a ground over some S, $\underline{H}$ is a ground over some T, $S \subseteq T$, and $\underline{\varphi}$ is a color-preserving map of graphs.

(2′) $\underline{G}$ is a ground over some S, $\underline{H}$ is a ground over some T, and there is an injective function $\alpha : S \hookrightarrow T$ such that for each edge $e \in E(\underline{G})$, $\mathrm{color}(\underline{\varphi}(e)) = \alpha(\mathrm{color}(e))$. (Should we call such $\underline{\varphi}$ a *color-injective* map of colored graphs?)

(3) Both $\underline{G}$ and $\underline{H}$ are grounds over some S, and $\underline{\varphi}$ is a color-preserving map of graphs.

(Definitions (2) and (2′) seem to be most reasonable.) *The kernel* $\mathrm{Ker}\,\underline{\varphi}$ of a *strong* ground homomorphism $\underline{\varphi} : \underline{G} \to \underline{H}$ is defined as a particular equivalence relation $\sim$ on $\underline{G}$ (both on vertices and on edges): $x \sim y \overset{\mathrm{def}}{\Leftrightarrow} \underline{\varphi}(x) = \underline{\varphi}(y)$. *The kernel* $\mathrm{Ker}\,\underline{\varphi}$ of a ground homomorphism $\underline{\varphi} : \underline{G} \dashrightarrow \underline{H}$: *on vertices* it can be defined similarly as the above equivalence relation $\sim$. Can $\sim$ be extended to edges? Consider two definitions:

(1) $e \sim e' \overset{\mathrm{def}}{\Leftrightarrow} \mathrm{color}(e) = \mathrm{color}(e')$ and $\iota(e) = \iota(e')$.

(2) $e \sim e' \overset{\mathrm{def}}{\Leftrightarrow} \mathrm{color}(e) = \mathrm{color}(e')$ and $\tau(e) = \tau(e')$.

Prove the equivalence of the two definitions. In the special case $pr : \underline{F}(a,b) \twoheadrightarrow \underline{F}(a)^{+b}$ we have the strong isomorphism $\underline{F}(a,b)/\!\sim\, \cong \underline{F}(a)^{+b}$. This is the ground analog of the first isomorphism theorem. Kernels of ground homomorphisms vs. transfer-invariant subgrounds (=normal subgrounds). *The image* $\mathrm{Im}\,\underline{\varphi}$ of a ground homomorphism $\underline{\varphi} : \underline{G} \dashrightarrow \underline{H}$ is also defined as the equivalence relation on $\underline{H}$ described by the following (constructive, geometric) gluing process: identify each pair of vertices in $\underline{\varphi}(V(\underline{G}))$, then perform foldings. Image *is* a subground of $\underline{H}$. The relation between transfer-invariant subgrounds (= normal subgrounds) on grounds, and loop-homogeneous graphs.

The description of *the amalgamated free product* of grounds, $\underline{A} *_{\underline{C}} \underline{B}$. Canonical representatives in free grounds and in free products. (For groups, the amalgamated free product is usually associated with *injective* group homomorphisms, $C \hookrightarrow A$ and $C \hookrightarrow B$, so we make a similar assumption for grounds: use injective ground homomorphisms. But this is not absolutely necessary.) For the trivial ground $\underline{C} := \underline{1}$, $\underline{A} *_{\underline{1}} \underline{B} = \underline{A} * \underline{B}$. The description of the *trivial* HNN-extension $\underline{A}* = \underline{A}*_{\underline{1}}$ of *one* ground $\underline{A}$ (rather than one group): an extra color is necessary to build a tree-like structure from multiple copies of $\underline{A}$. $\underline{A}*$ happens to be equal to a free product of two grounds. Which ones? The description of a general HNN-extension $\underline{A}*_{\underline{C}}$.

For extra fun.

(1) The kernel $\sim$ of a ground homomorphism $\underline{\varphi} : \underline{G} \to \underline{H}$ was defined above abstractly: on vertices by $v \sim v' \overset{\mathrm{def}}{\Leftrightarrow} \underline{\varphi}(v) = \underline{\varphi}(v')$. Assuming that both $\underline{G}$ and $\underline{H}$ are given constructively as quotients of free grounds by some sets of paths, can that same

kernel also be described constructively and geometrically starting with one vertex and using the rewriting function that comes with the ground homomorphism $\underline{\varphi}$?

(2) We showed in the homework that each ground $\underline{G}$ over S naturally leads to a group, namely, $\mathrm{gr}(\underline{G}) := \mathsf{Paths}(\underline{G})/{\sim_{\underline{G}}}$ with the operation $\cdot$ as described in class (transfer plus concatenation). Describe in what sense each color in S can be interpreted as an element of the group $\mathrm{gr}(\underline{G})$. This gives a function $\gamma : S \to \mathrm{gr}(\underline{G})$. Then prove that $\gamma(S) \subseteq \mathrm{gr}(\underline{G})$ is a generating set of $\mathrm{gr}(\underline{G})$.