Math 525 Algebraic topology I. Spring 2024. Igor Mineyev
Homework, topics, and fun.

Below, "$*$" means "turn in", "no $*$" means "do not turn in, but know how to solve". If a text is in yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked "**for extra fun**" are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration. I will also include a list of topics.

- As part of the ongoing $\mathsf{IG^3OR'S}$ group, please feel free to join our weekly mini-seminar on Zoom in Spring 2024 (the meeting number and password were given in class). We meet each Tuesday at 9.30am. Xianhao An, Jihong Cai and Leslie Hu will present various articles by Akio Kawauchi that claim to have solved several long-standing open problems in topology/geometric group theory. Feel free to read the articles as well and try to find mistakes, if any.

Topics: Algebra and topology, metric space, topology, open sets, closed sets, topological space, examples of topological spaces, continuous function (= map), homeomorphism; constructing new topological spaces: subspace topology, the topology of disjoint union, product topology, quotient topology, saturated sets; balls and cubes in $\mathbb{R}^n$, disk D^n, boundary of a disk, manifold, sphere S^n, torus T^n, projective plane $\mathbb{R}P^2$ (3 definitions), projective space $\mathbb{R}P^2$, examples of surfaces, attaching map, cell complex (= CW-complex), weak topology, cell=open cell e_i^n, ...

Homework 1. Due Friday, January 26, Friday. Handwritten, stapled, due at the beginning of the class.

(1) Show that the open unit disc in $\mathbb{R}^n$ (= the interior of D^n) is homeomorphic to $\mathbb{R}^n$.
(2*) Prove that $\mathbb{R}\mathrm{P}^2$ is a manifold. (Do not forget "Hausdorff".)
(3*) Let X be the result of collapsing ∂D^2 in the disk D^2 to a point, with the quotient topology. Prove that X is homeomorphic to S^2.
(4*) Give three different definitions of the projective plane $\mathbb{R}\mathrm{P}^2$ (as in class). Prove that they give the same topological space (i.e. they are homeomorphic).
(5) Show that $(-\infty, 0]$ and $\mathbb{R}$ (with their usual topology) are not homeomorphic.
(6) The rest of problems have been moved to the next homework for the lack of time this week.

For extra fun:

- Give a reasonable definition of the topological spaces S^∞ and $\mathbb{R}\mathrm{P}^\infty$.

Topics: Each characteristic map Φ_i is continuous, cellular structures on S^n and $\mathbb{R}\mathrm{P}^n$, paths and loops in a topological space, homotopy of maps, path homotopy, path homotopy is an equivalence relation, concatenation of paths, loops, fundamental group, $\pi_1(\mathbb{R}^n)$, path-connected topological space, simply connected, change of basepoint, the fundamental group of a cartesian product (for *arbitrary* spaces), S^n is simply connected for $n \geq 2$, $\pi_1(S^1)$, a covering space, a lift of a path, existence of a lift of a path, existence of a lift of a path homotopy, invariance of π_1 under homeomorphisms (hw), connected topological space, connected component of a topological space, path-connected component, ...

Homework 2. Due on Friday, February 2.

(1) Section 1.1, The fundamental group: Basic constructions, p. 38: # 5*. (Make sure to turn in this problem since it is marked with "*". Note that π_1 is defined by homotopy

preserving endpoints, but a homotopy of maps $S^1 \to X$ is not required to preserve a basepoint.)

(2) Show that path homotopy is an equivalence relation on the set of paths in a topological space X.

(3) Prove that the fundamental group is homeomorphism-invariant, i.e. if (X, x_0) and (Y, y_0) are homeomorphic pairs, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

(4*) Prove that a subset A in a cell complex X (with the weak topology) is open (closed) in X if and only if for each characteristic map $\Phi_i^n : D_i^n \to X$, the set $(\Phi_i^n)^{-1}(A)$ is open (closed) in D_i^n. [See p. 519.]

(5*) Show that any path-connected topological space is connected. Show that if a topological space is connected and locally path-connected, then it is path-connected. (Hint: Use connected components and path-connected components.)

(6*) Prove that a cell complex is connected if and only if it is path connected. [Hint: First show that any cell complex is locally path connected. See p. 523.] More generally, show that for any cell complex X, its connected components and path components agree.

(7) Section 1.1, The fundamental group: Basic constructions, p. 38: # 2, 3, 10*, 11, 14*.

(8) Given topological spaces X and Y, prove that the standard projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

(9) Learn the proof that $\pi_1(S^1) \cong \mathbb{Z}$, p. 29-31.

For extra fun:

(1)  Describe, as precisely as possible, the fundamental group of the waste basket. (This one is from my office.) The same question for the *surface* of this wastebasket. What is the genus of this surface?

(2) Trace the definition of cell complex X to define the surjective function $\sqcup_{n,i} D_i^n \twoheadrightarrow X$. Deduce from the above exercises that the weak topology on X is the same as the quotient topology induced by this function.

Topics: Induced homomorphism, composition of induced homomorphisms, a retraction of X onto A, $r : X \twoheadrightarrow A$ and $r_A : X \to X$, a retract of X, no retraction from D^2 onto $\partial D^2 = S^1$, Brouwer fixed-point theorem, (strong) deformation retraction of X onto A, a deformation retract of X, deformation retraction implies isomorphism in π_1, Moebius band, homotopy equivalence, contractible space, invariance of π_1 under homotopy equivalence (p. 37), wedge of (pointed) topological spaces, the fundamental theorem of algebra, ...

Homework 3. Due on Friday, February 9.

(1*) Is the sphere S^2 homeomorphic to the torus T^2? Generalize to S^n and T^n for $n \geq 2$.

(2) Suppose $r : X \to Y$ is a retraction and $x_0 \in Y$. Show that the homomorphism r_* induced by r on the fundamental groups (at x_0) is surjective. If $\iota : Y \hookrightarrow X$ is the inclusion map, prove that the induced homomorphism ι_* is injective. (See p.36)

(3) Section 1.1, The fundamental group: Basic constructions, p. 38: # 16*, 18*.

(4) Prove that if Y is a deformation retract of X, then X and Y are homotopy equivalent.

(5) A topological space X is *contractible* if X is homotopy equivalent to the topological space $\{pt\}$ consisting of one point. Deduce that if X deformation retracts to a point, then it is contractible.

(6) About homotopy equivalence and deformation retractions, chapter 0, p. 18: #1, 2, 5*, 6a*, 6b*. [Hint for problem 5: use the product topology and compactness of the

interval $[0,1]$.] [The fact that homotopy equivalence is an equivalence relation can be used without proof if needed. This will be an exercise in the next homework.]

For extra fun:

- The Poincaré conjecture says that any closed simply connected 3-dimensional manifold is homeomorphic to S^3. Find and read its proof. (It is quite hard.) Give a different, shorter proof.
- Pick a particular cell complex X. Deform it until it is unrecognizable to obtain a cell complex Y. Prove that X is homotopy equivalent to Y. Repeat.

Topics: Free group is a group, reduced word over a family of groups $\{G_\alpha\}$, free product (associativity by representing by permutations, injective $L : W \to S_W$), the universal property of the free product, the kernel of a homomorphism, the first isomorphism theorem for groups, $i_\alpha : A_\alpha \to \cup_\alpha A_\alpha = X$, $j_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$, the van Kampen theorem (proof by switching designation between parts), relation to pushouts.

Homework 4. Due on Friday, February 16. There is discrepancy between the published textbook and the newer file for the textbook available online. Whenever this happens, always use the file.

(1) About homotopy equivalence, chapter 0, p. 18: # 3a*. Prove that homotopy equivalence is an equivalence relation. (It is on the class of topological spaces, not on a set.)

(2) Prove that if X and Y are path-connected topological spaces and $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for any choice of $x_0 \in X$. (See p. 28 and 37 for an arbitrary homotopy equivalence.)

(3) Learn the proof of the van Kampen theorem, p. 43-46. The main principle: switching from one part to another. Construct partitions either explicitly or using Lebesgue numbers.

(4*) Prove *in two ways* that the fundamental group of a finite connected graph X (= finite path-connected cell complex of dimension at most 1) is a free group. The first way: construct a homotopy equivalence between X and a wedge of finitely many circles. The second way: use the van Kampen theorem directly. (All this can actually be generalized to arbitrary connected graphs.)

(5*) A *closed surface* is a compact surface without boundary. Describe some examples (at least three) of surfaces that are compact and have boundary. Describe some examples (at least three) of surfaces that are not compact and have no boundary. For each of these examples, show that it is homotopy equivalent to a graph. What are the fundamental groups of these surfaces?

(6) The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 2, 3*, 4*, 7*, 16*. If you claim that a space is path-connected, provide reasoning.

For extra fun:

(a) Come up with a more geometric/topological proof of associativity of the free product of two groups $G * H$ using the wedge of two spaces $X \vee Y$, where $\pi_1(X) \cong G$ and $\pi_1(Y) \cong H$, and using the universal covering of $X \vee Y$. (It might be easier to first do this in the special case when both X and Y are wedges of circles.)

(b) Count the number of cells in each dimension of the standard cellulation of T^3. Then do this for T^n.

(c) Pick your favorite manifold, construct a cellular structure on it, count the number of cells in each dimension. Find most efficient cellulations of this manifold, or at least as efficient as possible.

(d) Do (b) and (c) for triangulations.

(e) A *handlebody* is an orientable surface embedded in $\mathbb{R}^3$ *together* with "its inside". A *Heegard decomposition* of a 3-manifold is its realization as a union of two copies of the same handlebody glued together along their boundary surfaces via some homeomorphism of the surface. Describe such a decomposition of S^3 into two 3-balls. Also a decomposition into two solid tori ($S^1 \times D^2$). Also into two pretzels (whose boundary is the surface of genus 2). What is the fundamental group of a handlebody? State clearly what the van Kampen theorem says for each of the above decompositions.

(f) Can one use this purely algebraic statement to prove the Poincaré conjecture? (The only proof known so far uses differential geometry.)

Topics: $A * 1 \cong A$, applications of the van Kampen theorem: $\pi_1(S^n)$ for $n \geq 2$ (again), the fundamental group of wedge sum, of a wedge of circles, attaching 2-cells to spaces, attaching n-cells for $n \geq 3$, the induced homomorphism of $X^{(2)} \hookrightarrow X$ (using properties of cell complexes below), fundamental groups of arbitrary complexes, group presentations, presentation complex, any group is a fundamental group, presentations of the fundamental groups of closed surfaces, connected sum, cell structures on surfaces (closed orientable of genus g, closed non-orientable of genus g), surfaces N_1 and N_2, distinguishing (homotopy types of) closed surfaces by orientation and genus, ...

Know before Exam 1 on Friday, February 23.

(1) The van Kampen theorem: applications to cell complexes, section 1.2, p. 52: # 8. If you claim that a space is path-connected, provide reasoning. [Hint for # 8: One can use the van Kampen theorem here, but it is easier to use cartesian products.]

For extra fun:

- **Knots.** *A knot* is a smooth or piecewise linear embedding of the circle S^1 into $\mathbb{R}^3$. Denote K the image of such an embedding. By the fundamental group of a knot K we mean the fundamental group of the knot complement, $\pi_1(\mathbb{R}^3 \setminus K)$. Describe in detail *the Wirtinger presentation* for the fundamental group of any knot K.
(See exercise 22, page 55.)
- Define the notion of an *orientation* on a manifold. For Riemannian manifolds, this can be done using the riemannian structure. For triangulated manifolds, use the simplicial structure. For topological manifolds, use relative singular homology.

Topics: Cell (=open cell), subcomplex (two definitions and their equivalence), subcomplex is a complex (use hw), compact subsets of cell complexes, finite complex, constructing neighborhoods in cell complexes, cell complexes are Hausdorff (and even normal), Two definitions of a subcomplex, the closure of an n-cell in $X^{(n)}$ equals the image of the characteristic map, compact subsets of cell complexes, covering spaces, a lift of a map (to a covering space), path lifting property, homotopy lifting property, homomorphism induced by a covering, covering spaces and subgroups (relation between topology and group theory, subgroups of free groups), locally path-connected spaces, lifting criterion for groups, asphericity of S^1.

Homework 5. Due on Friday, March 1. Starting from this homework, the assignments will be due on Learn@Illinois: look for "MATH 525 F1 SP24: Algebraic Topology I (Mineyev, I)". Please write by hand, either on paper or on a pad. Submit any time before the beginning of the class on Friday.

(1) Let A be a subcomplex of a cell complex X. (With the first definition of a subcomplex.) Prove inductively that the topology on $A \cap X^{(n)}$ induced from X and the topology induced on $A \cap X^{(n)}$ by inductively attaching cells of dimensions $0, 1, 2, \ldots, n$ are the same. (p.520. This is used to prove that a subcomplex is a complex.)

(2) Prove that any cell complex is Hausdorff (with respect to the weak topology). See p. 522, fill out details.

(3) Show that for any n-cell e^n in a cell complex X, the closure of e^n in $X^{(n)}$ is the same as the closure of e^n in X.

(4) Show that for any cell complex X and any n, $X^{(n)}$ is closed in X. Deduce that $X^{(n)}$ is a subcomplex of X.

(5*) Show that abelianization homomorphisms $\alpha_G : G \to G_{ab}$ commute with quotient homomorphisms in the following sense. If G is a group, R is a subset of G and $\langle\langle R \rangle\rangle_G$ is the subgroup of G normally generated by R, denote $H := G/\langle\langle R \rangle\rangle_G$ and let $q_R : G \to H$ be the quotient homomorphism. Then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, two heads, "\alpha_G"] \arrow[d, "q_R"] & G_{ab} \arrow[d, "q_{\alpha_G(R)}"] \\ H \arrow[r, two heads, "\alpha_H"] & H_{ab} \end{tikzcd}$$

$q_{\alpha_G(R)}$ here means the quotient map of G_{ab} by the subgroup normally generated by the subset $\alpha_G(R) \subseteq G_{ab}$. (Since G_{ab} is abelian, "normally generated" is the same as "generated".) This was used to distinguish surfaces of different genera (orientable and nonorientable). (Here is a more precise definition of the function $q_{\alpha_G(R)}$. It is the function induced by q_R. Specifically, for each $y \in G_{ab}$, take any $x \in G$ such that $\alpha_G(x) = y$. Then let $q_{\alpha_G(R)}(y) := \alpha_H \circ q_R(x)$. Check that this is a well-defined function, and is a homomorphism. Then check that the kernel of $q_{\alpha_G(R)}$ is the normal subgroup of G_{ab} normally generated by the set $\alpha_G(R)$.)

(6*) Let F_n be the free group of rank n with basis $\{x_1, \ldots, x_n\}$. Prove that an element $x_{i_1}^{m_1} \ldots x_{i_k}^{m_k}$ of F_n, where each x_{i_j} is an element of the basis, belongs to the commutator subgroup F_n' if and only if, for each $i \in \{1, \ldots, n\}$, the sum of exponents of x_i occurring in $x_{i_1}^{m_1} \ldots x_{i_k}^{m_k}$ is zero. (Hint: Use the identity $ba[a^{-1}, b^{-1}] = ab$.)

(7*) Use (6) to show that the abelianization of F_n is $\mathbb{Z}^n$.

(8) Describe how (5) and (7) are useful for computing abelianizations of groups that are given by presentations.

For extra fun:

- What would you mean by a *cubical cell complex*? Try to define it. Look up for a precise definition. What would it mean for a cubical complex to be (locally or globally) positively curved?
- Given an orientable surface M, how would you associate, in a natural way, a cubical complex to it, with an action by $\pi_1(M)$ on it?
- Similarly, given a hyperbolic 3-manifold M, how would you associate a cubical complex to it, with an action by $\pi_1(M)$?
- Many problems in 3-dimensional manifold theory have been solved relatively recently using cubical complexes. The virtual fibering conjecture, etc. How would you use cubical complexes to do it?

Topics: Lifting criterion (for spaces), the unique lifting property, universal covering, semi{locally simply connected} space, existence of universal coverings (hw), existence of a covering for a given subgroup of $\pi_1(X)$.

Homework 6. Due on Friday, March 8.

(1) Learn the proof of the homotopy lifting property for arbitrary covering spaces. It is the same as in the proof of the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$.
(2) Learn the proof of the existence and uniqueness of the universal covering space.
(3*) Give an explicit description of a covering space of the wedge of two circles, $S^1 \vee S^1$, that is contractible. Describe the covering projection map and prove that it is indeed a covering space.
(4) Find the universal covering of the projective plane $\mathbb{R}\mathrm{P}^2$. Generalize to $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$.
(5) Covering spaces, section 1.3, p.79: # 1, 2*, 3, 4*, 7*, 9*.
(6) Learn the proof of the classification theorem for covers, p. 67.
(7) Let $p : \tilde{X} \to X$ be a covering space, Y be a path-connected space, $f : Y \to X$ be a constant map. Prove that any lift of f to $\tilde{X}$ is a constant map.

For extra fun:

- Pick a particular cell complex X, for example a finite graph. Construct a covering $\tilde{X}$ of this complex. Find a generating set for $\pi_1(\tilde{X})$ and describe it in terms of loops in $\tilde{X}$. Construct another cover of X, find a generating set. Repeat.
- Which 2-dimensional complexes are aspherical?
- Be the first one to prove or disprove the Whitehead conjecture: any subcomplex of any aspherical 2-dimensional cell complex is aspherical. (See also the articles by Kawauchi; either confirm the proof or find a mistake.)

Topics: Maps of covering spaces, isomorphism of covering spaces, uniqueness of covering spaces (for a given subgroup), the classification of covering spaces, uniqueness of a universal covering space, regular covering (= normal covering), deck transformation (= automorphism of a covering space), $G(\tilde{X})$, group actions, the action of $G(\tilde{X})$ on $\tilde{X}$, a regular cover, the characterization of regular covers, $G(\tilde{X}) \cong N(H)/H$, Cayley graph (hw), deck transformations are uniquely determined by (their value at) one point (i.e. the action is free), orbit space (=quotient space), regular covers arising from group actions, subgroups of free groups (Schreier subgroup theorem or Nielsen-Schreier subgroup theorem).

Homework 7. Due on Friday, March 29, before 2 p.m.

(1) Find all the deck transformations for the coverings in problems (3) and (4) in the previous homework.
(2*) Let $\langle S|R\rangle$ be a presentation of a group G. Consider the following two graphs (= 1-dimensional cell complexes) $\mathcal{G}$ and $\mathcal{G}'$.
   (a) The *Cayley graph* for the generating set S of G is the graph $\mathcal{G}$ whose vertices v_g one-to-one correspond to the elements $g \in G$ and edges $e_{g,s}$ one-to-one correspond to the elements $(g,s) \in G \times S$; the left end of each edge e_g is attached to v_g and the right end to v_{gs}.
   (b) Let $X_{S,R}$ be the presentation complex for the presentation $\langle S|R\rangle$ (that is one vertex, one edge for each $s \in S$, and one 2-cell for each $r \in R$), and let $\mathcal{G}'$ be the 1-skeleton of the universal cover $\tilde{X}_{S,R}$ of $X_{S,R}$.

We know that $\pi_1(X_{S,R}) \cong G$. Prove that the graphs $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic (or, equivalently, homeomorphic as cell complexes). [Hint: First extend the Cayley graph $\mathcal{G}$ to the *Cayley complex*, see page 77. Then use the uniqueness of universal covers.]

(3) Covering spaces, section 1.3, p.79: # 14*, 16*, 17*, 18*, 19* (only first two parts). (In 16, if X is empty and Y is not empty, then one can construct an example such that $X \to Z$ is normal, but $Y \to Z$ is not normal. Also, if we allow Y to not be path-connected, one also can construct an example such that $X \to Z$ is normal, but $Y \to Z$ is not normal. To fix this – and for simplicity – assume in exercise 16 that each of the spaces X, Y, Z, is nonempty and path-connected. In 16, the assumption of being locally path-connected does seem to be necessary. It should be used to show that the image of a basic neighborhood in X is exactly some basic neighborhood in Y.
In 17, make sure to describe X and $\tilde{X}$, and check the rest of conditions.
In 18, it is not explained precisely what is meant by saying that "one covering of X is a covering of another covering of X". Here is the precise definition: given two covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ of the same space X, we say that p_1 *is a covering of* p_2 if there is a covering map $r : X_1 \to X_2$ such that $p_1 = p_2 \circ r$. (This can be restated as commutativity of the diagram with p_1, p_2 and r.)
In 19, prove only the first two parts, skip the last sentence about embedding of M_g to T^3.)

(4) *Let X be a path-connected, locally path-connected, and semilocally simply-connected space with a basepoint x_0, and $H \leq \pi_1(X, x_0)$. Then there exists a covering space with a basepoint, $p_H : (X_H, x_{H0}) \to (X, x_0)$, that realizes H, that is, $p_{H*}(\pi_1(X_H, x_{H0})) = H$.*

The proof of this in the textbook is a bit sketchy, a somewhat more detailed proof is outlined below. Fill in the details.

First construct a universal cover as before, that is,

$$\tilde{X} := \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}.$$

Define a relation $\sim_H$ on $\tilde{X}$ by

$$[\gamma] \sim_H [\gamma'] \quad \Leftrightarrow \quad \gamma(1) = \gamma'(1) \text{ and } [\gamma\overline{\gamma'}] \in H.$$

(One can check that this equivalence relation can be equivalently described by the orbits of a specific H-action on $\tilde{X}$, by lifting paths, that will be described below.) Check that $\sim_H$ is an equivalence relation and let $X_H := \tilde{X}/\sim_H$. An element in X_H is an equivalence class $[[\gamma]]_H$. If c is the constant path at x_0 in X, we let $\tilde{x}_0 := [c] \in \tilde{X}$ and $x_{H0} := [[c]]_H \in X_H$ be the basepoints.

Define the projections

$$(\tilde{X}, \tilde{x}_0) \xrightarrow{\tilde{p}} (X_H, x_{H0}) \xrightarrow{p_H} (X, x_0)$$

by $\tilde{p}([\gamma]) := [[\gamma]]_H$ and $p_H([[\gamma]]) := \gamma(1)$, check that they are well-defined and their composition is p. Prove that $\tilde{p}$ and p_H are covering spaces. The main point is that for any two basic neigborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ (path-connected, from the construction of $\tilde{X}$), their projections to X_H, $\tilde{p}(U_{[\gamma]})$ and $\tilde{p}(U_{[\gamma']})$, are either disjoint or coincide. The space X_H (together with the projections $\tilde{p}$ and p_H) is called *an intermediate cover* for the cover p.

Next check that the covering space $p_H : (X_H, x_{H0}) \to (X, x_0)$ indeed realizes the subgroup H, that is, $p_{H*}(\pi_1(X_H, x_{H0})) = H$. This is proved by the following argument.

Let γ be any loop in X at x_0, γ_H be its lift at x_{H0} in X_H, and $\tilde{\gamma}$ be the lift of γ_H at $\tilde{x}_0$ in $\tilde{X}$. Directly construct a lift $\tilde{\gamma}'$ of γ starting at $\tilde{x}_0$ in $\tilde{X}$ as follows. For $t \in [0,1]$ let $\gamma_t : [0,1] \to X$ be the initial part of γ reparameterized linearly by the unit interval $[0,1]$. Let $\tilde{\gamma}'(t) := [\gamma_t] \in \tilde{X}$. ($\tilde{\gamma}'$ is a path, not necessarily a loop.) Check that $\tilde{\gamma}'$ is a lift of γ that starts at $\tilde{x_0}$, then by the uniqueness of lifts, $\tilde{\gamma}' = \tilde{\gamma}$. In particular, $\tilde{\gamma}'$ is a lift of γ_H.

The equality $p_{H*}(\pi_1(X_H, x_{H0})) = H$ follows from the equivalences (prove them): for any element $[\gamma] \in \pi_1(X, x_0)$,

$$
\begin{aligned}
&[\gamma] \in p_{H*}(\pi_1(X_H, x_{H0})) \\
\Leftrightarrow\ & \exists \text{ a loop } \gamma'_H \text{ in } X_H \text{ at } x_{H0} \text{ such that } \gamma \text{ is path-homotopic to the loop } p_H \circ \gamma'_H \\
\Leftrightarrow\ & \text{the lift of } \gamma \text{ at } x_{H0} \text{ in } X_H \text{ is a loop} \\
\Leftrightarrow\ & \gamma_H \text{ is a loop (at } x_{H0} \text{ in } X_H) \\
\Leftrightarrow\ & \gamma_H(0) = \gamma_H(1) \qquad (\text{since } \tilde{\gamma}' \text{ is a lift of } \gamma_H) \\
\Leftrightarrow\ & \tilde{p} \circ \tilde{\gamma}'(0) = \tilde{p} \circ \tilde{\gamma}'(1) \quad \Leftrightarrow \quad \tilde{p}([\gamma_0]) = \tilde{p}([\gamma_1]) \\
\Leftrightarrow\ & \tilde{p}([c]) = \tilde{p}([\gamma]) \quad \Leftrightarrow \quad [[c]]_H = [[\gamma]]_H \\
\Leftrightarrow\ & [\gamma] \sim_H [c] \quad \Leftrightarrow \quad [\gamma \bar{c}] \in H \quad \Leftrightarrow \quad [\gamma] \in H.
\end{aligned}
$$

(5) The description of the group of deck transformations for any path-connected covering (Proposition 1.39, p. 71):
Let X be a path-connected, locally path-connected space, $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected covering, and let $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$. Then:
 (a) *The covering p is regular $\Leftrightarrow$ the subgroup H is normal in $\pi_1(X, x_0)$.*
 (b) *The group of deck transformations $G(\tilde{X}) = G(\tilde{X}, p)$ is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

The proof in the book is confusing and convoluted, a more structured proof is provided below.

Proof. (a) Since X is locally path-connected, then $\tilde{X}$ is locally path-connected as well. Prove the following equivalences:

the covering p is regular (wrt definition 1) (since $\tilde{X}$ is path-connected)

$\Leftrightarrow$ the covering p is regular wrt definition 2

$\Leftrightarrow \forall\, \tilde{x}, \tilde{x}' \in p^{-1}(x_0)\;\; \exists\, g \in G(\tilde{X})$ such that $g(\tilde{x}) = \tilde{x}'$

$\Leftrightarrow \forall\, \tilde{x}_1 \in p^{-1}(x_0)\;\; \exists\, g \in G(\tilde{X})$ such that $g(\tilde{x}_0) = \tilde{x}_1$

(by the corollary of the lifting criterion, Prop. 1.37, p. 67, proved in class)

$\Leftrightarrow \forall\, \tilde{x}_1 \in p^{-1}(x_0)\;\; p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

$\Leftrightarrow \forall\, \tilde{x}_1 \in p^{-1}(x_0)\;\; \forall$ path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ $\;\; p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*([\tilde{\gamma}]^{-1}\pi_1(\tilde{X}, \tilde{x}_0)[\tilde{\gamma}])$

$\Leftrightarrow \forall\, \tilde{x}_1 \in p^{-1}(x_0)\;\; \forall$ path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ $\;\; H = [p \circ \tilde{\gamma}]^{-1} H [p \circ \tilde{\gamma}]$

(by path-lifting, since $\tilde{X}$ is path-connected)

$\Leftrightarrow \forall$ loop γ at x_0 in $X\;\; H = [\gamma]^{-1} H [\gamma]$

$\Leftrightarrow \forall\, g \in \pi_1(\tilde{X}, x_0)\;\; H = g^{-1} H g$

$\Leftrightarrow H$ is normal in $\pi_1(X, x_0)$.

(b) Define $\varphi : N(H) \to G(\tilde{X})$ by lifting loops as follows. For each $[\gamma] \in N(H) \leq \pi_1(X, x_0)$, lift the loop γ to a path $\tilde{\gamma}$ at $\tilde{x}_0$ in $\tilde{X}$, then let $\varphi([\gamma])$ be the deck transformation of $\tilde{X}$ sending $\tilde{x}_0 = \tilde{\gamma}(0)$ to $\tilde{x}_1 := \tilde{\gamma}(1)$. Such a deck transformation exists, because $p(\tilde{x}_1) = p(\tilde{\gamma}(1)) = \gamma(1) = x_0$, so $\tilde{x}_1 \in p^{-1}(x_0)$, and as above,

$$[\gamma] \in N(H) \;\Leftrightarrow\; H = [\gamma]^{-1} H [\gamma] \;\Leftrightarrow\; H = [p \circ \tilde{\gamma}]^{-1} H [p \circ \tilde{\gamma}]$$

$$\Leftrightarrow\; p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*([\tilde{\gamma}]^{-1}\pi_1(\tilde{X}, \tilde{x}_0)[\tilde{\gamma}])$$

$$\Leftrightarrow\; p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \quad \text{(by the corollary of the lifting criterion)}$$

$$\Leftrightarrow\; \exists\, g \in G(\tilde{X}) \text{ such that } g(\tilde{x}_0) = \tilde{x}_1.$$

For each $[\gamma] \in N(H)$, such a deck transformation g is unique because the endpoint $\tilde{\gamma}(1)$ is independent of the choice of representative γ in the path-homotopy class $[\gamma]$ in X, and by the uniqueness of lifting property (since $\tilde{X}$ is connected).

To show that φ is surjective, for any $g \in G(\tilde{X})$ first choose a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1 := g(\tilde{x}_0)$, then let γ be its projection, that is, $\gamma := p \circ \tilde{\gamma}$. Since g is the deck transformation sending $\tilde{x}_0$ to $\tilde{x}_1 = g(\tilde{x}_0)$, the same equivalences above show that $[\gamma] \in N(H)$. Since g is the deck transformation sending $\tilde{x}_0 = \tilde{\gamma}(0)$ to $\tilde{x}_1 = \tilde{\gamma}(1)$, then $\varphi([\gamma]) = g$, which proves surjectivity of φ.

Next prove that φ is a homomorphism. Given any $[\gamma], [\gamma'] \in N(H)$, by the definition of φ,

- $\tau := \varphi([\gamma])$ is the deck transformation sending $\tilde{x}_0$ to $\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of γ at $\tilde{x}_0$ in $\tilde{X}$,
- $\tau' := \varphi([\gamma'])$ is the deck transformation sending $\tilde{x}_0$ to $\tilde{\gamma}'(1)$, where $\tilde{\gamma}'$ is the lift of γ' at $\tilde{x}_0$ in $\tilde{X}$,
- $\tau'' := \varphi([\gamma][\gamma']) = \varphi([\gamma \cdot \gamma'])$ is the deck transformation sending $\tilde{x}_0$ to $\tilde{\gamma}''(1)$, where $\tilde{\gamma}''$ is the lift of the concatenation $\gamma \cdot \gamma'$ at $\tilde{x}_0$ in $\tilde{X}$.

We want to show that $\tau'' = \tau \circ \tau'$. The path $\tilde{\gamma}''$ is a lift of $\gamma \cdot \gamma'$, and the concatenation $\tilde{\gamma} \cdot (\tau \circ \tilde{\gamma}')$ is also a lift of $\gamma \cdot \gamma'$ because

$$p \circ (\tilde{\gamma} \cdot (\tau \circ \tilde{\gamma}')) = (p \circ \tilde{\gamma}) \cdot (p \circ \tau \circ \tilde{\gamma}') = (p \circ \tilde{\gamma}) \cdot (p \circ \tilde{\gamma}') = p \circ (\tilde{\gamma} \cdot \tilde{\gamma}') = \gamma \cdot \gamma'.$$

The two lifts start at the same point $\tilde{x}_0$, hence by the uniqueness of lifts, $\gamma'' = \tilde{\gamma} \cdot (\tau \circ \tilde{\gamma}')$. The deck transformation τ'' sends $\tilde{x}_0$ to $\tilde{\gamma''}(1) = \tilde{\gamma} \cdot (\tau \circ \tilde{\gamma}')(1) = (\tau \circ \tilde{\gamma}')(1) = \tau(\tilde{\gamma}'(1))$. The deck transformation $\tau \circ \tau'$ also sends $\tilde{x}_0$ to $\tau(\tilde{\gamma}'(1))$, because τ' sends $\tilde{x}_0$ to $\gamma'(1)$ and τ sends $\gamma'(1)$ to $\tau(\gamma'(1))$. By the uniqueness of lifts (since $\tilde{X}$ is connected), we conclude that $\tau'' = \tau \circ \tau'$, so φ is a homomorphism.

Finally, we describe the kernel of φ: for any $[\gamma] \in N(H) \leq \pi_1(X, x_0)$,

$$\begin{aligned} [\gamma] \in \operatorname{Ker} \varphi \; &\Leftrightarrow \; \varphi([\gamma]) = id \\ &\Leftrightarrow \; \text{the lift of } \gamma \text{ at } \tilde{x}_0 \text{ in } \tilde{X} \text{ is a loop} \\ &\Leftrightarrow \; [\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \; \Leftrightarrow \; [\gamma] \in H. \end{aligned}$$

Therefore, the lifting of loops induces an isomorphism $G(\tilde{X}) \cong N(H)/H$. □

(6) We emphasize that, by the proof above, the action of $N(H)/H$ on $\tilde{X}$ is defined by lifting loops to paths at $\tilde{x}_0$.
If, in addition, $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a regular covering, then $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$. If, in addition, p is a universal covering, then $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)$.
In both cases, $\pi_1(X, x_0)$ acts on $\tilde{X}$ by lifting loops to paths.

For extra fun:

- Provide a more conceptual proof of the existence of a covering $(X_H, x_{H0}) \to (X, x_0)$ for any given subgroup $H \leq \pi_1(X, x_0)$. First construct the universal covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, then define an action of H on $\tilde{X}$ by homeomorphisms using lifts of paths. Prove that the space $X_H := \tilde{X}/{\sim_H}$ constructed in (4) is the same as the quotient of $\tilde{X}$ by the H-action. That is, the equivalence classes in $\tilde{X}$ are the same as the H-orbits in $\tilde{X}$.

Topics: A triangulation of a manifold, simplex, face (two interpretations: set and map, face maps $\Delta^{n-1} \to \Delta^n$), restriction to a face, Δ-complex, each Δ-complex gives rise to a (particular) cell complex (without proof), simplicial chain, simplicial complex, taking the boundary of a manifold twice, the standard open simplex, an open simplex in a Δ-complex, the chain complex $C_*^{simp}(X) := \Delta_*(X) := \mathbb{Z}[\Sigma_*^{simp}(X)]$, $\partial \circ \partial = 0$, simplicial homology $H_n^{\Delta}(X)$ of a Δ-complex, a chain complex, the chain complex $C_*(X) = C_*^{sing}(X)$, singular homology $H_n(X) = H_n^{sing}(X)$ of a topological space X, cycles and boundaries, homology of a point (hw), homology of a disjoint union.

Homework. To know before Exam 2 on Friday, April 5.

(1) What is the fundamental group of the projective plane $\mathbb{R}P^2$? Prove the answer. What is the fundamental group of the projective plane $\mathbb{R}P^n$ for $n \geq 1$?
(2) Learn examples 2.2-2.5 for simplicial homology, pp. 106-107.

Topics: Chain map f_* between chain complexes, $f \rightsquigarrow f_* \rightsquigarrow f_{**}$, exact sequence, short exact sequence, the long exact sequence induced by a short exact sequence of chain complexes (hw), the chain complex for a pair (X, A), $C_*(X, A)$, the short exact sequence of chain complexes for any pair (X, A), relative homology $H_*(X, A)$, the corresponding long exact sequence (involving $H_*(A)$, $H_*(X)$, $H_*(X, A)$), the reduced singular homology $\tilde{H}_*(X)$ for nonempty X

(similarly reduced simplicial homology), reduced relative homology $\tilde{H}_*(X, A)$ for nonempty A, $\tilde{H}_*(X, A) \cong H_*(X, A)$ for nonempty A, the long exact sequence for *reduced* relative homology (involving $\tilde{H}_*(A)$, $\tilde{H}_*(X)$, $\tilde{H}_*(X, A)$), *one* long exact sequence for a good pair (X, A) (only for reduced homology $\tilde{H}_*(A)$, $\tilde{H}_*(X)$, $\tilde{H}_*(X/A)$; proof will come later after excision), subcomplexes form good pairs (without proof), chain homotopy between chain maps, homotopy induces chain homotopy, $f \sim g \Rightarrow f_* \sim g_* \Rightarrow f_{**} = g_{**}$ (the maps induced on singular homology by homotopic maps coincide).

Homework 8. Due on Friday, April 12.

(1) Compute the 0th (singular) homology of any topological space X, $H_0(X) = H_0(X; \mathbb{Z})$. (See Proposition 2.7, p. 109.)

(2) Prove that for nonempty X, $H_0^{sing}(X) \cong \tilde{H}_0^{sing}(X) \oplus \mathbb{Z}$. (The first isomorphism theorem is not enough here.)

(3*) Give a full proof that each short exact sequence of chain complexes

$$0 \to A_* \to B_* \to C_* \to 0$$

gives rise to a long exact sequence of homology groups $H_n(A_*)$, $H_n(B_*)$, $H_n(C_*)$. (The best is not to look in the book, at least first; do it first as an exercise on your own. In the book this is explained on pages 116–117.)

(4) Compute the (singular) homology groups of a point. Compute the reduced homology groups of a point.

(5) Simplicial and singular homology, section 2.1, p. 131: #4, 5*, 7*. (For # 7 make an educated guess how to glue the 3-simplex to obtain the 3-sphere, without proof. Then compute simplicial homology. For extra fun: Prove that the result of gluing is indeed *topologically* the 3-sphere. Also, is the result of this (combinatorial) gluing a Δ-complex? Is it a simplicial complex?)

Topics: Chain homotopy equivalence of chain complexes, $X \sim Y \Rightarrow C_*(X) \sim C_*(Y) \Rightarrow H_*(X) \cong H_*(Y)$ (homotopy invariance of singular homology), $H_n(\sqcup_i X_i) \cong \oplus_i H_n(X_i)$, homology and path components (not for reduced homology),
four applications of long exact sequences: $H_n(X, x_0) \cong \tilde{H}_n(X, x_0) \cong \tilde{H}_n(X)$, homology of spheres (reduced and otherwise), no retraction from D^n to ∂D^n, the Brouwer fixed-point theorem for D^n;
the degree of a map $S^n \to S^n$, the statement of the excision theorem (for relative singular homology).

Homework 9. Due on Friday, April 19.

(1) Learn the rest of the proof that homotopy leads to chain homotopy (prism operator) and then to isomorphism in homology, $f \sim g \Rightarrow f_* \sim g_* \Rightarrow f_{**} = g_{**}$. See Theorem 2.10, pp. 111-112.

(2) Simplicial and singular homology, section 2.1, p. 131: #11*, 12*, 13*, 23.

(3*) Write a full proof that for any path connected topological space X, $H_1(X)$ is (isomorphic to) the abelianization of $\pi_1(X)$. (Try to do this without looking in the book first. There is a proof on pp. 166-167.) This statement is part of what is known as the Hurewicz theorem.

Topics: The cone-off map $b : LC_n(Y) \to LC_{n+1}(Y)$ (for $b \in Y$, Y convex in $\mathbb{R}^k$), the barycenter of a linear simplex, the barycentric subdivision of a linear simplex, the proof of the excision theorem, $H_*^{\mathcal{U}}(X) \cong H_*(X)$ (i.e. $H_*(X)$ is isomorphic to the homology using arbitrarily small singular simplices in X), the barycentric subdivision of a linear chain (in $LC_n(Y)$), the barycentric subdivision of a singular chain (in $C_n(X)$), the cone-off map $b : LC_n(Y) \to LC_{n+1}(Y)$ is a chain homotopy between id_* and 0_* (this is *not* called a contracting homotopy), the subdivision maps $S_* : LC_*(Y) \to LC_*(Y)$ and $S_* : C_*(X) \to C_*(X)$ are chain maps, the chain homotopy T_* between S_* and $id_* : LC_*(Y) \to LC_*(Y)$, the chain homotopy T_* between S_* and $id_* : C_*(X) \to C_*(X)$, the chain homotopy between S_*^m and id_*, subdivision of linear n-simplices shrinks the diameter by $\frac{n}{n+1}$, the Lebesque number of an open cover (of a compact metric space), the long exact sequence of a triple (X, A, B) (involving $H_*(A, B)$, $H_*(X, B)$, $H_*(X, A)$; none for reduced homology), the proof of $H_n(X, A) \cong \tilde{H}_n(X/A)$ for good pairs, $H_i(D^n, \partial D^n) \cong \tilde{H}_i(D^n, \partial D^n) \cong \tilde{H}_i(S^n)$, the proof of the long exact sequence for a good pair (X, A) (involving $\tilde{H}_n(X/A)$).

Homework 10. Due on Friday, April 26.

(1) Learn the remaining details of the proof of the excision theorem, pp.119-124.

(2*) How to compute the homology of any wedge sum $\vee_{j\in J} X_j$ for good pairs (X_j, x_j): write a detailed proof of Corollary 2.25, p. 126. (You can use Proposition 2.22 that we will discuss in class). Then use it to compute the reduced homology of any wedge of n-spheres, $\vee_{j\in J} S_j$. That is, each S_j is homeomorphic to the n-sphere S^n for the same n, and the problem is to compute $\tilde{H}_i(\vee_{j\in J} S_j)$ in each dimension i. (One way to prove this is to use Proposition 2.22 directly: apply it to the pair $(X, A) := (\sqcup_j X_j, \sqcup_j \{x_j\})$. In this case, you would need to come up with an explicit isomorphism $\oplus_j H_n(X_j, \{x_j\}) \cong H_n(\sqcup_j X_j, \sqcup_j \{x_j\})$ for all n. Check that it is induced by (post-composition with) the inclusions $X_{j'} \hookrightarrow \sqcup_j X_j$. Another way is to use the long exact sequence for the pair $(X, A) = (\sqcup_j X_j, \sqcup_j \{x_j\})$, and then replace $H_n(X, A)$ with $\tilde{H}_n(X/A)$ using Proposition 2.22. This way works only for $n \geq 2$. Proving this for all $n \geq 2$ is enough for the purpose of the homework.)

(3) Simplicial and singular homology, section 2.1, p. 131: # 22*. (This problem deals with *singular* homology. In this problem, "free" should be interpreted as "free abelian". You can use the result from algebra: any subgroup of a free abelian group G is free abelian, and of rank at most the rank of G. Hint for part (b): first use long exact sequences to show that the maps in the nth homology induced by inclusions $X^n = X^{n+1} \hookrightarrow X^{n+2} \hookrightarrow \ldots$ are isomorphisms, $H_n(X^n) = H_n(X^{n+1}) \cong H_n(X^{n+2}) \cong \ldots$, and use this to prove that $H_n(X) \cong H_n(X^n)$. For extra fun: Part (b) actually holds even if X is not finite-dimensional. Can you prove part (b) for any cell complex X?)

(4) Give a detailed computation of the cellular homology of (particular cellular structures on) closed surfaces. (See particular cellular structures in examples 2.36 and 2.37, p. 141. Use the cellular boundary formula, p.140.)

For extra fun:

- Suppose X is a simplicial complex, then we can view it as a cell complex, and therefore have two notions of homology for X: simplicial and singular. What is the relation between these two homologies?
  Since each simplicial complex can be viewed as a cell complex, and simplicial homology can be viewed as cellular homology (check that the boundary maps are indeed the same in the two cases), then we can ask a more general question: for any cell complex X, what is the relation between the singular homology and the cellular homology of X? It turns out, they are isomorphic. The best proof is the one using the tool of homological

algebra called *spectral sequences.* Figure it out. Another proof is on pp. 139-140 in the textbook, using the 5-lemma.

- Can one comb a hairy sphere? Try to comb the hair on a sphere in such a way that no hair stands up. Is this possible? See how degrees of maps of spheres can be used to solve this problem, p. 135 in the textbook.
- Is there a version of Mayer-Vietoris sequence co compute the (singular) homology of X when it is covered by several (more than two) subsets? The answer is what is called the Mayer-Vietoris *spectral sequence*, which is more involved than a long exact sequence. Make a statement and prove it.
- We defined singular homology with coefficients in an abelian group, or more generally, any module M over any ring R, denoted $H_n(X; M)$.
  Let X be any path-connected cell complex X and $G := \pi_1(X)$. Determine how the (singular) homology of X with coefficients in the group ring $\mathbb{Z}G$ can be described in terms of the universal cover $\tilde{X}$. The same question for cellular homology.

Topics: Homology of wedge sum (hw), the 5-lemma (hw), $H_n^{simp}(X) \cong H_n^{sing}(X)$ (without proof), invariance of simplicial homology under homeomorphisms (and under homotopy equivalence), definition of cellular homology $H_n^{cell}(X)$, $H_n^{cell}(X) \cong H_n^{sing}(X)$ (without proof), invariance of cellular homology under homeomorphisms (and under homotopy equivalence), the cellular boundary formula (without proof), Betti numbers, the Euler characteristic of a finite cell complex, invariance of the Euler characteristic under homeomorphisms, homology with coefficients (in abelian groups or modules), the Mayer-Vietoris sequence (without proof).

Homework. To know before the final exam.

(1) Know the statement of the 5-lemma, see p.129.
(2) Know the statement of the Mayer-Vietoris sequence, see p. 149.

For extra fun:

- Prove the 5-lemma by diagram chasing, without looking in the textbook.
- Deduce the Mayer-Vietoris sequence as the long exact sequence corresponding to a particular short exact sequence of chain complexes.

- **The seminar talk.** I encourage you to attend the GGT seminar generally. See the master calendar of all seminars. I am giving a talk at GGT seminar on Thursday, May 2, 2024, at 11am, Altgeld Hall 143: *The topology and geometry of units and zero-divisors: origami.* It relates several areas of mathematics. Not required, but feel free to come if you are interested in those topics.
- **ColorTaiko!** On the same day, on May 2, from 2pm to 5pm there is a poster session for Illinois Mathematics Lab that will include a poster for the "ColorTaiko!" computer game. (Preliminary version for now.) This IML project is based on my paper *The topology and geometry of units and zero-divisors: origami.*
- **The evaluation forms for this class.** The online evaluation forms should be available some time, probably between April 19 and May 1, 2024. You either receive this information by email or directly log in on the website `https://go.illinois.edu/ices-online` . I very much encourage you to fill out the evaluation forms. Since there is a deadline, please don't miss it. Also, bring your electronic devices at the beginning of the last class on Wednesday to fill out registration forms online.

- **Possible projects for summer and after.** If you are interested in doing a project with me over the summer, please let me know. (Either tell me in class, or at office hours, or call me by phone.)