Math 503. Introduction to geometric group theory. Spring 2024. Igor Mineyev
Homework.

Below, "$*$" means "turn in", "no $*$" means "do not turn in, but know how to solve". If a text is in the yellow color, the homework is still at a preliminary stage and might be modified later, but feel free to start working on it. The problems marked "**for extra fun**" are some interesting related problems; they will not affect your grade for the course, but should be good sources of inspiration.

- Take a look at
`https://mineyev.web.illinois.edu/class/24s/503/` and
`https://mineyev.web.illinois.edu/mineyev-ggt.html` .
Discuss with me any of the topics.

- As part of the ongoing IG³OR′S group, please feel free to join our weekly mini-seminar on Zoom in Spring 2024 (the meeting number and password were given in class). We plan to meet each Thursday, starting from January 25, at 9.30am. Xianhao An, Jihong Cai and Leslie Hu will read and present several articles by Akio Kawauchi that claim to have solved several long-standing open problems in topology/geometric group theory. Feel free to read the articles as well and try to find mistakes, if any.

Week 1 (two days): What is geometric group theory, relation to other areas, two ways to obtain a group, groups and spaces, the importance of group actions, Cayley graph, group action on $\mathcal{G}$.
Week 2: Cayley graphs and covering spaces, free groups (four definitions), the Cayley graph of a free group, the standard basis, a basis (two definitions), reduced edge path in a graph (combinatorial), combinatorial fundamental group of a graph (hw), each group is a quotient of a free group, the (?) rank of a free group, how to represent subgroups of free groups, {covering spaces of a graph} $\leftrightarrow$ {subgroups of a free group}, ...
Week 3: Immersions of graphs, Stallings folding, the Nielsen theorem, the statement of the Nielsen-Schreier theorem, representing Nielsen transformations by diagrams, the proof of the Nielsen theorem, ...
Homework 1. Due at the beginning of the class on Friday, February 2. Hand-written, stapled.

(1*) Let S be a set. Prove that the first definition of the free group $F = F(S)$ with basis S (using reduced words) indeed gives a group. That is, check that the operation is well-defined and the group axioms hold. (Hint: There are tedious algebraic ways to prove this, but a combinatorial/geometric group theory approach is much better. Even though $F(S)$ is a priori is not known to be a group, it might be helpful firsts to construct what *should be* its Cayley graph. The question is *not* to give the definition of the Cayley graph of F. The question is to construct a graph first *without using the group F*, and then to use that graph to prove associativity, etc. The action of $F(S)$ on this graph is not absolutely necessary, but you can define it if you like.)

(2*) Do the same for the second definition of $F(S)$ (using equivalence classes). Let S be a set. Prove that the second definition of the free group with basis S
  (a) gives a well-defined multiplication operation,
  (b) indeed gives a group, and
  (c) agrees with the first definition, that is, gives an isomorphic group.

  (Again, it is much better to give a proof that is more in line with geometric group theory. Define what *should be* the Cayley graph of $F(S)$, then use it to prove that the multiplication operation is well-defined, associativity, etc.)

(3*) Prove that the first and the second definitions of $F(S)$ imply the third one (universal property). (Do not forget to prove both existence and uniqueness.)

(4*) Prove that the Nielsen transformation $(a, b) \mapsto (a, ba)$ induces an isomorphism $F(a, b) \to F(a, b)$. (Hint: One way is to use the universal property.)

Week 4: Finish the proof of the Nielsen theorem, Nielsen's algorithm.
Corollaries of Nielsen' s algorithm: subgroups of free groups are free, rank is well defined, generators of $Aut(F_n)$, F_n is hopfian.
For free groups finitely generated ⇔ finite rank, seeing rank from a graph, the Howson theorem, the Hanna Neumann theorem, the Hanna Neumann conjecture (HNC), reduced rank (for a free group and for a graph), trees and flowers, Euler characteristic, essential and inessential edges, maximal essential set.

For extra fun:

- Generalize Nielsen's argument to show that the rank of a free group is well-defined *for any cardinality*, that is, the cardinality of any two bases of $F(S)$ is the same. This is equivalent to saying that if $F(S) \cong F(T)$, then there is a bijection $S \to T$.

Week 5: The group of automorphisms of F_n, generators of $Aut(F_n)$ (algorithmic), the Strengthened Hanna Neumann conjecture (SHNC), fiber product (= pull-back), reduced rank, the Euler characteristic and rank (only for connected graphs), system of graphs (or of complexes), leafage, order-essential edges, relation to ℓ^2-cohomology, free product $A * B$, the Marshall Hall theorem.
Week 6: The proof of the Marshall Hall theorem (using arrow tips), F_n-action on its Cayley graph (by preconcatenation), ping-pong lemma, free group ⇔ free action on a tree (two proofs: a fundamental domain or a maximal subtree), another proof of the Nielsen-Schreier theorem, group presentations, presentation complex, closed orientable surfaces, models of the hyperbolic plane $\mathbb{H}^2$, tessellations of $\mathbb{H}^2$, hyperbolic structures on closed orientable surfaces. (recursive construction of the universal covering space and the Cayley complex, one-relator, small-cancellation, hyperbolic), why surface groups (or, more generally, word hyperbolic groups) usually have many free subgroups.
Homework 2. Due Friday 22, 2024.

(1) (not to turn) Let $\mathcal{G}$ be the Cayley graph of a group G with respect to some generating set $S \subseteq G$ and w be a word in the alphabet $S \sqcup S^{-1}$. Prove that w represents the trivial element in G if and only if any edge path in $\mathcal{G}$ labeled w is a loop.

(2*) Prove that if $U \subseteq F_n$ is a finite generating set of F_n and $|U| = n$, then U is a basis of F_n.

(3*) Let X be any graph. (Always assumed to be oriented.) Write out the definition of the *combinatorial fundamental group* of X, $\pi_1^{comb}(X)$, as was indicated in class (using reduced edge paths). Define the operation, and prove that $\pi_1^{comb}(X)$ is a free group. (Hint: Prove that there is a maximal subtree in any *connected* graph. Compare this graph to the graph obtained by collapsing a maximal subtree to a vertex.)

(4*) Let X and Y be finite graphs and $Y \to X$ be a (combinatorial) covering space. Prove that the image of the induced homomorphism $\pi_1^{comb}(Y) \to \pi_1^{comb}(X)$ is a subgroup of finite index in $\pi_1^{comb}(X)$. (For the definition of a combinatorial covering space, use "locally bijective" instead of "locally injective". This exercise is helpful for Stallings' approach to the proof of the Marshall Hall theorem. If you like, do this exercise first in the case when X is *a rose*, that is, a graph with one vertex.)

(5*) Let Y be a graph with a basepoint vertex y_0, Y_1 and Y_2 be subgraphs such that $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2$ is a tree containing y_0. Prove that $\pi_1^{comb}(Y) \cong \pi_1^{comb}(Y_1) * \pi_1^{comb}(Y_2)$. (Again, this is useful for the Marshall Hall theorem.)

For extra fun:

- **Knots.** *A knot* is a smooth or piecewise linear embedding of the circle S^1 into $\mathbb{R}^3$. Denote K the image of such an embedding. By the fundamental group of a knot K we mean the fundamental group of the knot complement, $\pi_1(\mathbb{R}^3 \setminus K)$. Describe in detail *the Wirtinger presentation* for the fundamental group of any knot K.
  See, for example exercise 22, page 55, in Hatcher's book "Algebraic topology", `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`

Week 7: A proof of left orderability for countable free groups using the ping-pong lemma, decision problems in groups, word problem for F_n, membership problem for f.g. subgroups in F_n, the Novikov-Boone theorem.

Week 8: $N \subseteq \mathbb{N}$ is recursive $\Leftrightarrow$ N and $\mathbb{N} \setminus N$ are recursively enumerable, unsolvability of many algorithmic problems in f.p. groups, recursive presentation, f.g. subgroup H of a recursively presented group G is recursively presented, the group H_N for a recursively enumerable set N, the Higman embedding theorem, a universal f.p. group.

Week 9: Free product with amalgamation $A *_C B$, HNN-extension $A*_C$, topological interpretation: cell complex for $A *_C B$ from the complexes X_A and X_B, cell complex for $A*_C$ from the complex X_A, the tree-like structure of the universal cover, $A*_1 \cong A * \mathbb{Z}$, relations to actions on trees, graphs of groups, a splitting of a group, a splitting over a finite subgroup, ends of groups, the Freudenthal-Hopf theorem, invariance of the number of ends under a change of generators.

The project. If you would like to participate in a project in this class studying and presenting some interesting topics related to geometric group theory (meaning, related to mathematics), please let me know. If you like, this can be counted as one additional homework.

For extra fun.

- Prove that $A * B$ can be equivalently defined as the group given by the presentation $\langle S, S' | R, R' \rangle$, where $\langle S|R \rangle$ and $\langle S'|R' \rangle$ are presentations for A and B, respectively.

- Let G be a finitely generated group and H be a subgroup of G of finite index. Prove that G is finitely generated if and only if H is finitely generated.
- Prove that the combinatorial fundamental group $\pi_1^{comb}(X, x_0)$ of a graph X, as defined in class and in homework, is isomorphic to the fundamental group $\pi_1(X, x_0)$ as defined in algebraic topology, when X is viewed as a topological space.

Homework 3. Due Friday, March 29, 2024.

(1) (not to turn) Prove that if $S = S_1 \sqcup S_2$, then $F(S) \cong F(S_1) * F(S_2)$.

(2*) Let S be a finite generating subset of a group G, and $\mathcal{G}$ be its Cayley graph with respect to S. Give an upper bound on the number of vertices in any ball of radius $r \in \mathbb{N}$ centered at a vertex in $\mathcal{G}$. (The distance is the word metric, or the path metric in $\mathcal{G}$.)

(3*) Solve the conjugacy problem for (F_n, S_1) where S_1 is some generating set of F_n. That is, describe an algorithm that, given words u, v over S_1, decides in finite time whether there exists $g \in F_n$ such that $u = gvg^{-1}$. [Hint. First reduce the problem to the standard basis S, algorithmically. Given words u, v over S and $g \in F_n$, such that $u = gvg^{-1}$, let g_t be the terminal segment of g of length t. It is convenient to draw these as paths in the standard Cayley graph of F_n. Find an upper bound on the length of $g_t v g_t^{-1}$, in terms of the lengths of u and v. Then prove that if u and v are conjugate in F_n, then they are conjugate by a reasonably short element g.]

(4*) Usually, the free product $A * B$ is defined formally algebraically (as was done in class). Then one needs to define the multiplication operation on $A * B$, show that it is well-defined, and prove that $A * B$ is indeed a group. Do this in the philosophy of geometric/combinatorial group theory. First define $A * B$ using reduced words. Before knowing that $A * B$ is a group, define a tree $\mathcal{T}$ *without using reduced words*, then show that the vertices (or edges) in $\mathcal{T}$ one-to-one correspond to reduced words, which in turn, one-to-one correspond to the reduced edge-paths from the identity vertex to each of the vertices. Show that the set of vertices will naturally split into two types corresponding to the groups A and B, each vertex of type A has $|A|$ outgoing edges, and each vertex of type B has $|B|$ incoming edges (and no other incident edges). Define the operation on $A * B$. Use the one-to-one correspondence between group elements and paths in the tree to prove that $A * B$ is a group.

(5*) Next, describe the above tree explicitly in the case $A := \mathbb{Z}$, $B := \mathbb{Z}$, that is, for the free product $\mathbb{Z} * \mathbb{Z} \cong F(a) * F(b) \cong F(a, b)$.

(6*) Prove that $A * B$ can be equivalently defined using equivalence classes, where equivalence is defined by reductions and their inverses. (Again, try to give a geometric proof rather than an algebraic one.)

(7*) Define a natural action of $A * B$ on the tree $\mathcal{T}$. Prove that the stabilizers of vertices in this tree are conjugates of A and B in $A * B$. What are the stabilizers of edges?

(8) (not to turn) Prove that if $\hat{X}$ is a tree and $\hat{\alpha} : \hat{Y} \to \hat{X}$ is an immersion, then $\hat{\alpha}$ is a leafage, meaning that the restriction of $\hat{\alpha}$ to each connected component of $\hat{Y}$ (= a *leaf*) is injective.

For extra fun.

- Generalize the construction of a tree in the homework to construct a tree $\mathcal{T}$ corresponding to a free product with amalgamation, $G = A *_C B$. [Hint: Use left cosets in A/C instead of elements in A, and use left cosets B/C' instead of elements in B.]
- Construct an action of $G = A *_C B$ on the tree $\mathcal{T}$, and describe the stabilizers of vertices and the stabilizers of edges.
- Provide a similar construction of a tree $\mathcal{T}$ corresponding to the HNN extension $G = A*_1$ over the trivial group. Define an action of G on $\mathcal{T}$.
- Generalize the construction of the tree $\mathcal{T}$ to arbitrary HNN-extensions $G = A*_C$. Define an action of $G = A*_C$ and describe the stabilizers of vertices and edges in $\mathcal{T}$.
- If G is a finitely generated group and $H \leq G$ is a subgroup of finite index.
- Prove that H and G are quasiisometric. Deduce that they have the same number of ends: $e(H) = e(G)$.

Week 10: Lipschitz maps, quasiisometry (two definitions), invariance of the number of ends under quasiisometries, quasiisometry invariants, splittings of groups, the Stallings structure theorem, the Grushko theorem, $\mathbb{R}^2$ and $\mathbb{H}^2$, hyperbolic spaces and groups, slim triangles, thin triangles.

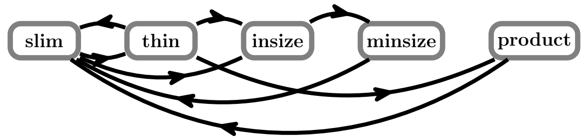
For extra fun.

- Using geometric group theory (for example, Cayley graphs), prove that $F(a,b)/\langle\,\langle[a,b]\rangle\,\rangle$ is exactly the abelianization of $F(a,b)$, that is, $\langle\,\langle[a,b]\rangle\,\rangle = [F(a,b), F(a,b)]$, the commutant of $F(a,b)$.
- Also prove that the group given by the presentation $\langle a, b \mid [a,b]\rangle$ is the free abelian group $\mathbb{Z} \times \mathbb{Z}$.

Week 11: Insize, minsize, Gromov product, equivalent definitions of hyperbolic groups hyperbolic groups are finitely presented, Dehn's algorithm, the solvability of word problem for hyperbolic groups, van Kampen diagrams for a presentation, isoperimetric inequalities, subquadratic isoperimetric inequality.

For extra fun.

- Read the proof of many equivalent definitions of hyperbolicity, and other properties of hyperbolic groups in the article
  [Alonso, Brady, Cooper, Ferlini, Lustig, Mihalik, Shapiro, Short, Notes on word hyperbolic groups]. (Either search and find it or use the address
  `http://math.hunter.cuny.edu/olgak/hyperbolic%20groups/MSRInotes2004.pdf`.
  (This is part of the book [Group theory from a geometrical viewpoint].)
  Write a detailed proof of the statement that the five definitions of hyperbolic graphs are equivalent: "slim triangles", "thin triangles", "insize", "minsize", "product", according to the following diagram:



The implications on the top are obvious, the ones on the bottom are not so much. Drawing pictures will be helpful.

- Prove that if a group G is hyperbolic, then it cannot contain a subgroup isomorphic to $\mathbb{Z}^2$. (This is an algebraic statement analogous to saying that standard hyperbolic spaces $\mathbb{H}^n$ cannot contain isometrically embedded flats, i.e. copes of the euclidean plane $\mathbb{E}^2$. See Proposition 3.5 and Proposition 3.6 in the above MSRI notes.)
- If G is a finitely generated group that has no subgroup isomorphic to $\mathbb{Z}^2$, does it follow that G is hyperbolic? (See the article by Brady given in class.)
- We proved in class that hyperbolicity for groups implies Dehn's algorithm, and hence a linear isoperimetric inequality (LIPI, for an appropriately chosen finite presentation). Prove that the following statements are equivalent for a finitely generated group G:
  - G is hyperbolic.
  - G is finitely presented and the finite presentation satisfies a linear isoperimetric inequality.
  - G is finitely presented and the finite presentation satisfies a subquadratic isoperimetric inequality.

  For this it will be helpful to read the article
  [A. Yu. Ol'shanskii, Hyperbolicity of groups with subquadratic isoperimetric inequality]
  `https://www.worldscientific.com/doi/abs/10.1142/S0218196791000183`
  (then click on "view article").

Week 12: Growth, the Gromov theorem on groups of polynomial growth, Dehn's function, isoperimetric inequalities, the ideal boundary of a hyperbolic group (two definitions), topology, conformal structures on the ideal boundary, the Cannon conjecture and conformal structures, homological descriptions of hyperbolic groups, quasiconvexity, abelian subgroups of hyperbolic groups, hyperbolicity and existence of $\mathbb{Z}^2$ subgroups ([ABCFLMSS], [Brady]), cubical complexes, Tietze transformations, the Tietze theorem.

Homework 4. Due Friday, April 12, 2024.

(1*) Prove that being quasiisometric is an equivalence relation on the class of all metric spaces. (The easiest is to use the first definition of quasiisometry given in class.)

(2) Prove the equivalence of the two definitions of quasiisometry given in class.

(3) Prove that the number of ends is a quasiisometry invariant, for locally finite graphs and for finitely generated groups.

(4*) Prove that free groups of rank m and n, F_n and F_m, are quasiisometric for any integers $m, n \geq 2$. (Hint: Prove that F_n can be realized as a finite-index subgroup of F_2. Observe that index can be defined using either right cosets or left cosets.)

(5*) Show that being finitely presentable is a quasiisometry invariant. More precisely, let G_1 and G_2 be finitely generated groups, with finite generating sets S_1 and S_2, respectively, and assume that G_1 and G_2 are quasiisometric. Prove that G_1 admits a finite presentation $\langle S_1 | R_1 \rangle$ if and only if G_2 admits a finite presentation $\langle S_2 | R_2 \rangle$. (Hint: First "connect the dots" by sending edges in one Cayley graph to short paths in the other Cayley graph. Then repeat this in the opposite direction. The label of an edge-path in a Cayley graph is a word in generators. Use the fact that in a locally finite Cayley graph there are only finitely many labels of loops of a given length.)

(6*) Prove that for finitely presented groups, having a linear isoperimetric inequality is a quasiisometry invariant. More precisely, let G_1 and G_2 be finitely presented groups that are quasiisometric. If $\langle S_1|R_1\rangle$ and $\langle S_2|R_2\rangle$ are some finite presentations for G_1 and G_2, respectively, prove that $\langle S_1|R_1\rangle$ satisfies a linear isoperimetric inequality if an only if $\langle S_2|R_2\rangle$ does. (Do not use equivalence to hyperbolicity, give a direct argument by drawing diagrams in the Cayley graphs. Those are images of van Kampen diagrams in the Cayley graphs.)

(7*) Prove the same statement where "linear" is replaced with "polynomial of degree $n \geq 1$".

For extra fun.

(1) Learn about boundaries of hyperbolic spaces and hyperbolic groups. Read and understand the following articles. Find the similarities and explain the differences among several presentations.

- Read Chapter 7 *The boundary of a hyperbolic space* of the book [Ghys-de la Harpe, Sur les groupes hyperboliques de M. Gromov]. Here is the english translation of that book: `http://perso.ens-lyon.fr/ghys/articles/groupeshyperboliques-english.pdf`
- [Alonso, Brady, Cooper, Ferlini, Lustig, Mihalik, Shapiro, Short, Notes on word hyperbolic groups]. (Either search and find it or use the address http://math.hunter.cuny.edu/olgak/hyperbolic%20groups/MSRInotes2004.pdf.
- Here is more on a precise conformal structure on the boundary of a hyperbolic group:
  - [Igor Mineyev, Flows and joins of metric spaces] (This paper constructs the metric $\hat{d}$ on the Cayley graph $\mathcal{G}$) `https://msp.org/gt/2005/9-1/p13.xhtml` Or see a preprint version at my website: `https://mineyev.web.illinois.edu/`
  - [Igor Mineyev, Metric conformal structures and hyperbolic dimension] (This paper constructs the metric $\check{d}$ on $\partial G = \partial\mathcal{G}$): `https://www.ams.org/journals/ecgd/2007-11-11/S1088-4173-07-00165-8/home.html` (And click on "Full-text PDF". Or see a preprint version at my website: `https://mineyev.web.illinois.edu/`)
- Read a review [I. Kapovich, N. Benakli, Boundaries of hyperbolic groups]: `https://arxiv.org/pdf/math/0202286.pdf`

Week 13: The Andrews-Curtis conjecture, properly discontinuous cocompact actions: relation between groups and metric spaces (Švarc-Milnor lemma [Bridson-Haefliger, p.140]).

For extra fun:

- Prove or disprove the Andrews-Curtis conjecture. (Either way, this is an extra credit problem!)

Week 14: CAT(0) and CAT(-1) spaces and complexes, are hyperbolic groups CAT(0)? (see also the symmetric join in "Flows and joins of metric spaces"), M_κ-complexes, $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, small cancellation groups, small cancellations and Dehn's algorithm, small cancellations and hyperbolicity.

Homework 5. Due Friday, April 26, 2024.

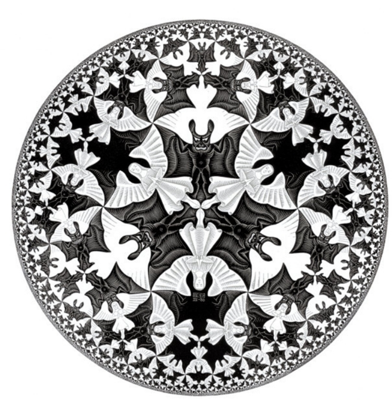
(1*) Solve the conjugacy problem for any hyperbolic group G with respect to some finite generating set S. That is, describe an algorithm that, given words u, v over S, decides in finite time whether there exists $g \in G$ such that $u = gvg^{-1}$. (We assume that a hyperbolicity constant δ is given as part of the initial data. Also, in order to solve the conjugacy problem you might need first to solve the word problem, for hyperbolic groups.)

(2*) Prove that quasigeodesics stay uniformly close to geodesics in any hyperbolic metric space X, and vice versa. This means that for each pair $(\lambda, K) \in (0,\infty)\times[1,\infty)$ there exists $R \in [0,\infty)$ such that for any pair $x, y \in X$, any geodesic p from x to y and any (λ, K)-quasigeodesic q from x to y, (the image of) p lies inside the R-neighborhood of (the image of) q, and vise versa. (See, for example, the above MSRI notes [ABCFLMSS, Notes on word hyperbolic groups, Chapter 3] or [Bridson-Haefliger book, pp. 401-405].)

(3*) Prove that hyperbolicity of groups is a quasiisometry invariant. That is, if G_1 and G_2 are finitely generated groups and G_1 is hyperbolic, then G_2 is hyperbolic.

(4*) Prove that the ideal boundary of a hyperbolic group is a quasiisometry invariant, at least as a set. That is, if G_1 and G_2 are quasiisometric hyperbolic groups, then there is a bijection $\partial G_1 \to \partial G_2$. Feel free to use the definition of ∂G either using Gromov products or using geodesic rays. (First you might need to prove that quasigeodesics rays in a locally finite Cayley graph stay close to geodesic rays. With some more work one can also show that this natural bijection is a homeomorphism.)

Week 15 (two classes): Small cancellations and CAT(-1), small cancellations and CAT(0), cubical complexes, local-to-global conditions, flag complexes give rise to CAT(0) cubical complexes, automatic groups, geodesic flow and symmetric join, conformal action, cross product and double difference, stereographic projection and flat Earth, mapping class groups, ...



- **The seminar talk.** I encourage you to attend the GGT seminar generally. See the master calendar of all seminars. I am giving a talk at GGT seminar on Thursday, May 2, 2024, at 11am, Altgeld Hall 143: *The topology and geometry of units and zero-divisors: origami*. It relates several areas of mathematics. Not required, but feel free to come if you are interested in those topics.
- **ColorTaiko!** On the same day, on May 2, from 2pm to 5pm there is a poster session for Illinois Mathematics Lab that will include a poster for the "ColorTaiko!" computer game. (Preliminary version for now.) This IML project is based on my paper *The topology and geometry of units and zero-divisors: origami*.
- **The evaluation forms for this class.** The online evaluation forms should be available some time, probably between April 19 and May 1, 2024. You either receive this information by email or directly log in on the website `https://go.illinois.edu/ices-online` . I very much encourage you to fill out the evaluation forms. Since there is a deadline, please don't miss it. Also, bring your electronic devices at the beginning of the last class on Wednesday to fill out registration forms online.
- **The projects for the class.** We meet at 4.30 pm on Friday, May 3, 2024 at the regular class. (Will try to find an available room.)
- **Possible projects for summer and after.** If you are interested in doing a project with me over the summer, please let me know. (Either tell me in class, or at office hours, or call me by phone.)