THE TOPOLOGY AND GEOMETRY OF UNITS AND ZERO-DIVISORS: ORIGAMI

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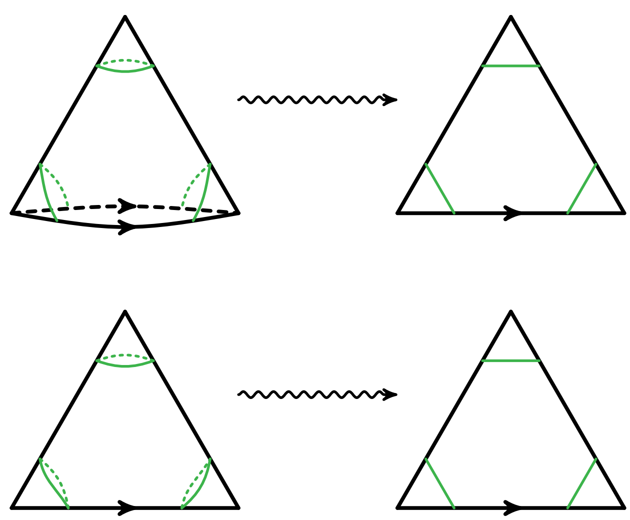
ABSTRACT. We define a product structure Π, its corresponding 2-dimensional cell complexes X_Π and Y_Π, associate to them the *universal groups* G_Π and $\bar{G}_\Pi$, and a pair (a_Π, b_Π) of elements in the group algebra $\mathbb{Z}_2\bar{G}_\Pi$ or in the group ring $R\bar{G}_\Pi$ for any ring R. We give lists of sufficient combinatorial conditions on a product structure Π implying that G_Π and $\bar{G}_\Pi$ are torsion-free and that the associated a_Π and b_Π are nontrivial units or zero-divisors. The proofs use graphs and geometry of cell complexes in a substantial way. These results allow using computer-based search to look for counterexamples to the Kaplansky unit and zero-divisor conjectures.



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Contents

1. INTRODUCTION.

The Kaplansky conjectures are several purely algebraic questions that have been open for a long time. In 1956-1957 Kaplansky presented a list of several questions about group algebras FG, where F is a field and G is a group (see [27], [28]). In this paper we will concentrate on two of those conjectures.

- *The unit conjecture.* For any torsion-free group G and any $a, b \in FG$, does $ab = 1$ imply that a and b are trivial units, meaning that a is a multiple of one element in G and b is a multiple of an element in G?
- *The zero-divisor conjecture.* For any torsion-free group G and any $a, b \in FG$, does $ab = 0$ imply that a and b are trivial zero-divisors, meaing that $a = 0$ or $b = 0$?

The unit conjecture has actually been known since 1940, when Higman stated it as a question in his article [26]. One counterexample to the unit conjecture is currently known, recently found by Gardam [22].

To study the Kaplansky conjectures, we propose a conceptual shift: instead of looking for nontrivial units or zero-divisors over a particular group G, we propose directing the effort to *constructing* multiple groups that would admit such units and zero-divisors. In section 2.1 we define *product structures* and *product substructures*. Those are abstract concepts – originally unrelated to any group – that reflect the structure of a desired unit or a zero-divisor. Next, sections 3.1, 3.2, 3.3 introduce topology into the picture: for any product structure or substructure Π we associate particular 2-dimensional complexes X_Π, Y_Π, $\bar{X}_\Pi$, $\bar{Y}_\Pi$. These complexes lead to *the full universal group* $\bar{G}_\Pi := \pi_1(\bar{X}_\Pi)$. We show in sections 3.4 and 3.5 how, given a product structure Π, one can associate specific units or zero-divisors a_Π and b_Π in the group rings $\mathbb{Z}_2\bar{G}_\Pi$ and $R\bar{G}_\Pi$, where R is any ring with unity.

While trying to construct counterexamples to the unit conjecture and to the zero-divisor conjecture, the main difficulty is to guarantee that the group $\bar{G}_\Pi$ is torsion-free and that the associated units and zero-divisors are nontrivial. We address these two challenges by using geometry of cell complexes: it is shown in sections 4.5 and 6.4 that a product structure Π is nondegenerate and the group $\bar{G}_\Pi$ is torsion-free *if* the corresponding compex Y_Π admits a metric structure of curvature ≤ 0. Then, as described in section 3.8, nondegeneracy implies the nontriviality of the associated units and zero-divisors.

Further, in section 7.1 we provide several purely combinatorial conditions on a product structure or substructure Π that guarantee the existence of such a metric structure on Y_Π. This allows for a unified computer search to look for counterexamples to the two Kaplansky conjectures. We finish with a general *program* on how to look for units and zero-divisors, in section 7.2.

This article combines five areas of mathematics: using topology and geometry, algebraic problems are translated into combinatorial questions about graphs that can be verified by computational means. It is the author's hope and belief that a computer search should be able to find examples satisfying the combinatorial conditions of section 7.1, and therefore provide multiple examples of groups with nontrivial units and zero-divisors, or that this general approach can be modified to enable finding counterexamples computationally. A negative computational result would also be of interest from a geometric viewpoint: if one checks computationally that for a given size (m, n) there are no product structures satisfying the combinatorial conditions, this shows that no 2-complexes Y_Π or $\bar{Y}_\Pi$ of this particular size (m, n) admit certain polyhedral metric structures of curvature ≤ 0.

The partition illustrated in Fig. 2 below is a result of the ongoing computational project in collaboration with Manisha Garg and Haizi Yu. The project is devoted to designing various algorithms and performing searches to look for product structures satisfying the combinatorial conditions described in section 7.1 below, and therefore, for counterexamples to the unit and zero-divisor conjectures. The methods and results of the computation will be published in a subsequent article.

Another outcome of this project is the author's ongoing joint activity with students at the University of Illinois to develop and keep improving the "ColorTaiko!" computer game, based on the taikos (= product graphs) defined in section 2.6 below. A player will draw pairs of edges in a bipartite graph, consecutively. The game will color the edges at each step and check whether certain combinatorial conditions are satisfied (as in section 7.1). The goal is to progress as much as possible towards creating a full partition of the edges in the complete bipartite graph. The goal of writing the game is to popularize Kaplansky conjectures to the general public and to engage students in research. Eventually, the "ColorTaiko!" game will be available to the public on the author's website and elsewhere.

Full disclosure: the research presented in this article is expressly *not* supported by the National Science Foundation. The author's proposal to write this article, to perform computational search for counterexamples to the Kaplansky conjectures and to develop the "ColorTaiko!" computer game was declined in February 2024 by the Topology program at the NSF. The reviewers and the panel did not count "the research itself" as "broader impact" contrary to the policies and procedures guide, stated – in spite of the existing Gardam counterexample to the unit conjecture – that "it would be helpful if the PI provided more context as to why they believe these conjectures to be false", that "the panel found the proposed research to be innovative, but speculative and would have liked to see more evidence that counterexamples would be found using this approach", that "some panelists had additional concerns that the outcomes would have minimal impact beyond the scope of the proposed problems", and that "the proposal would have been stronger if had more clearly addressed the potential societal outcomes that would result from the activities described".

2. Combinatorial notions describing units and zero-divisors

2.1. **Product structures and product substructures.** Let $\mathcal{G}(A, B)$ denote the complete bipartite graph on two finite sets A and B. The numbers m and n will always denote the cardinalities of A and B, respectively. We will identify the edges in $\mathcal{G}(A, B)$, which we will call *the vertical edges*, with the elements of the cartesian product $(a, b) \in A \times B$. *A product structure* is a triple $\Pi = (A, B, P)$ in which

- $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ are finite sets,
- P is a partition of the set of edges in $\mathcal{G}(A, B)$, that is, of the set $A \times B$, such that any two distinct edges $(a, b), (a', b') \in A \times B$ belonging to a cell of the partition P have no

common vertices, i.e.,

$$\forall C \in P\ \forall (a,b),(a',b') \in C \quad ((a,b) \neq (a',b') \Rightarrow (a \neq a' \text{ and } b \neq b')).$$

A product substructure is a triple $\Pi = (A, B, P)$ in which

- $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ are finite sets,
- P is *a subpartition* of the set $A \times B$, i.e., a family of subsets in $A \times B$ such that

$$\forall C_1, C_2 \in P \quad (C_1 \neq C_2 \Rightarrow C_1 \cap C_2 = \emptyset),$$
$$\forall C \in P\ \forall (a,b),(a',b') \in C \quad ((a,b) \neq (a',b') \Rightarrow (a \neq a' \text{ and } b \neq b')).$$

Clearly, each product structure is a product substructure. The pair (m, n) will be called *the size* of Π.

Let P be a partition or a subpartition. P will be called *even* if each cell in P has exactly two elements. We will call such cells *2-cells*. P will be called *odd* if it has exactly one cell having one element (*1-cell*) and all other cells are 2-cells.

For the purpose of studying zero-divisors in group algebras of the form $\mathbb{Z}_2 G$, Schweitzer [32, Definition 2.4] considered partitions of the set $\{(i,j) \mid i \in \{1,\dots,n\}, j \in \{1,\dots,m\}\}$ into pairs, and he defined the related *matched rectangles* as a means of illustrating such partitions. Our product structure notion is similar in spirit, but is more general since it applies both to zero-divisors and to units, and we additionally require that the two edges in each 2-cell have no common vertices.The corresponding notion of *taiko* (= the product graph) that we introduce in section 2.6 is a different way to illustrate units and zero-divisors; it is suitable for illuminating both the combinatorial and topological/geometric nature of our approach.

2.2. **Signatures.** *A signature* on a product structure (A, B, P) is a function $\sigma : A \sqcup B \to \{1, -1\}$ such that for each 2-cell $\{(a_i, b_j), (a_{i'}, b_{j'})\}$ of the partition P, $\sigma(a_{i'})\sigma(b_{j'}) = -\sigma(a_i)\sigma(b_j)$. Here one should think of $\{1, -1\}$ as being a subset of R, where R is any ring with unity 1. *A product structure with signature* is a quadruple $\Pi^{\pm} = (A, B, P, \sigma)$ such that (A, B, P) is a product structure and σ is a signature on (A, B, P).

2.3. **Group rings.** As the name suggests, a product structure is a tool describing the structure of products in group rings. $\mathbb{Z}_2$ will denote the field of order 2. Given any group G, each pair of elements in the group algebra $\mathbb{Z}_2 G$,

$$a = \sum_{i=1}^{m} a_i, \quad b = \sum_{j=1}^{n} b_j, \quad a_i, b_j \in G \subseteq \mathbb{Z}_2 G,$$

naturally leads to the product structure $(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\}, P)$, where P is the partition of $\{a_1, \dots, a_m\} \times \{b_1, \dots, b_n\}$ into the equivalence classes under the equivalence relation

$$(a_i, b_j) \sim (a_{i'}, b_{j'}) \quad \Leftrightarrow \quad a_i b_j = a_{i'} b_{j'}.$$

If, in addition, $ab = 1$ or $ab = 0$, then this partition P admits a refinement P' that is odd or even, respectively. Conversely, we will see in sections 3.4 and 3.5 below how certain product structures lead to groups $\bar{G}_\Pi$ and to certain associated elements a_Π and b_Π in group rings $\mathbb{Z}_2 \bar{G}_\Pi$ or $R\bar{G}_\Pi$. These elements are units if mn is odd and zero-divisors if mn is even.

The important and difficult questions are: when is the group $\bar{G}_\Pi$ torsion-free and when are a_Π and b_Π nontrivial? These properties are necessary to guarantee that a given product structure Π indeed leads to a counterexample to the unit or zero-divisor conjectures. We will address these questions in theorems 10 and 23 below, by topological and geometric means. First, let us present a convenient way of illustrating product structures.

Product structure example 1:
$\{(a_4,b_8),(a_5,b_5)\}$, $\{(a_6,b_4),(a_7,b_3)\}$, $\{(a_1,b_1),(a_2,b_9)\}$, $\{(a_6,b_5),(a_7,b_4)\}$, $\{(a_1,b_2),(a_2,b_1)\}$, $\{(a_5,b_6),(a_7,b_5)\}$, $\{(a_1,b_3),(a_3,b_1)\}$, $\{(a_4,b_9),(a_5,b_8)\}$, $\{(a_6,b_7),(a_7,b_6)\}$, $\{(a_1,b_4),(a_2,b_3)\}$, $\{(a_3,b_2),(a_4,b_1)\}$, $\{(a_6,b_8),(a_7,b_7)\}$, $\{(a_1,b_5),(a_2,b_4)\}$, $\{(a_4,b_2),(a_7,b_8)\}$, $\{(a_1,b_6),(a_3,b_4)\}$, $\{(a_4,b_3),(a_5,b_9)\}$, $\{(a_1,b_7),(a_2,b_6)\}$, $\{(a_3,b_5),(a_4,b_4)\}$, $\{(a_5,b_1),(a_6,b_9)\}$, $\{(a_1,b_8),(a_2,b_7)\}$, $\{(a_4,b_5),(a_5,b_2)\}$, $\{(a_6,b_1),(a_7,b_9)\}$, $\{(a_2,b_2),(a_3,b_9)\}$, $\{(a_5,b_7),(a_6,b_6)\}$, $\{(a_2,b_5),(a_3,b_3)\}$, $\{(a_2,b_8),(a_3,b_6)\}$, $\{(a_6,b_2),(a_7,b_1)\}$, $\{(a_3,b_7),(a_4,b_6)\}$, $\{(a_5,b_3),(a_7,b_2)\}$, $\{(a_3,b_8),(a_4,b_7)\}$, $\{(a_5,b_4),(a_6,b_3)\}$.

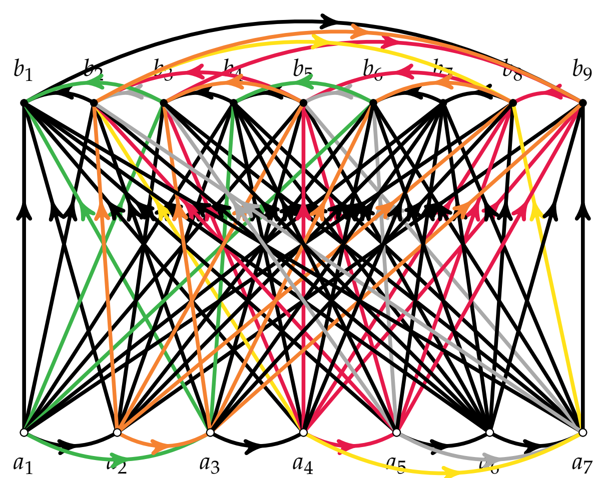


FIGURE 1. Size $(m,n) = (7,9)$. Taiko (the product graph). The 1-cell of the partition is $\{(a_1,b_9)\}$. The 31 2-cells of the partition are split into 6 colors. There are no folds. Some patterns repeat.

Product structure example 2:
$\{(a_1,b_1),(a_2,b_2)\}$, $\{(a_1,b_2),(a_2,b_3)\}$, $\{(a_2,b_1),(a_3,b_3)\}$, $\{(a_4,b_1),(a_1,b_3)\}$, $\{(a_3,b_1),(a_5,b_4)\}$, $\{(a_3,b_2),(a_4,b_4)\}$, $\{(a_1,b_4),(a_6,b_5)\}$, $\{(a_2,b_4),(a_7,b_3)\}$, $\{(a_3,b_4),(a_6,b_3)\}$, $\{(a_3,b_5),(a_4,b_2)\}$, $\{(a_4,b_3),(a_7,b_5)\}$, $\{(a_5,b_1),(a_1,b_5)\}$, $\{(a_5,b_2),(a_7,b_4)\}$, $\{(a_5,b_3),(a_8,b_5)\}$, $\{(a_8,b_1),(a_2,b_5)\}$, $\{(a_8,b_4),(a_4,b_5)\}$, $\{(a_5,b_5),(a_7,b_2)\}$, $\{(a_6,b_1),(a_8,b_2)\}$, $\{(a_6,b_2),(a_8,b_3)\}$, $\{(a_7,b_1),(a_6,b_4)\}$.

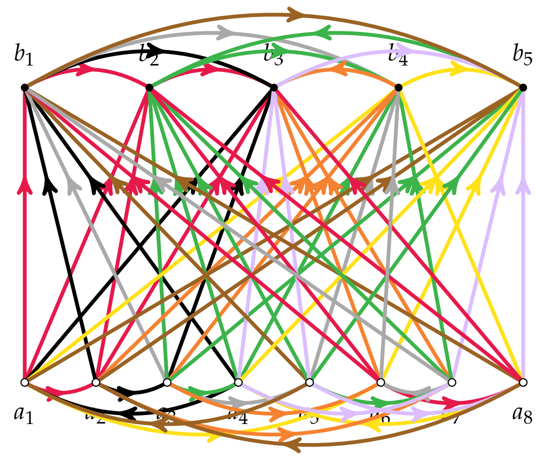


FIGURE 2. Size $(m, n) = (8, 5)$. Taiko (the product graph). The 20 2-cells of the partition are split into 8 colors. There are no folds. There are no repeating patterns. (What is the girth of the bottom graph L_A?)

2.4. **Horizontal edges.** Given a product structure or substructure Π, by *the horizontal edges* of Π we will mean the elements of the sets below; they come from the 2-cells in the partition P.

$$\begin{aligned}
\bar{E}_A &:= \{\{a, a'\} \in \bar{E}_A \mid \exists\, b, b' \in B \ \ \{(a, b), (a', b')\} \in P\},\\
\bar{E}_B &:= \{\{b, b'\} \in \bar{E}_B \mid \exists\, a, a' \in A \ \ \{(a, b), (a', b')\} \in P\},\\
\bar{E}_{AB} &:= \bar{E}_A \sqcup \bar{E}_B,\\
E_A &:= \{(a, a') \in \bar{E}_A \mid \exists\, b, b' \in B \ \ \{(a, b), (a', b')\} \in P\},\\
E_B &:= \{(b, b') \in \bar{E}_B \mid \exists\, a, a' \in A \ \ \{(a, b), (a', b')\} \in P\},\\
E_{AB} &:= E_A \sqcup E_B.
\end{aligned}$$

An orientation on Π is a function $O : \bar{E}_{AB} \to E_{AB}$ such that

- for each $\{a, a'\} \in \bar{E}_A$, $O(\{a, a'\}) = (a, a')$ or $O(\{a, a'\}) = (a', a)$,
- for each $\{b, b'\} \in \bar{E}_B$, $O(\{b, b'\}) = (b, b')$ or $O(\{b, b'\}) = (b', b)$, and
- for each 2-cell $\{(a, b), (a', b')\} \in P$,

$$(O(\{a, a'\}) = (a, a') \text{ and } O(\{b, b'\}) = (b, b')) \text{ or}$$
$$(O(\{a, a'\}) = (a', a) \text{ and } O(\{b, b'\}) = (b', b)).$$

We will say that a product substructure Π is *orientable*, or that Π satisfies the orientation condition, if there exists an orientation on Π.

2.5. **Bottom graph L_A, top graph L_B.** Given a subproduct structure $\Pi = (A, B, P)$ in which P is either even or odd, *the bottom graph* L_A is the unoriented graph whose vertex set is A

and the set of edges is $\bar{E}_{\Pi A}$. *The top graph* L_B is the unoriented graph whose vertex set is B and the set of edges is $\bar{E}_{\Pi B}$. Denote $\mathsf{L}_{AB} := \mathsf{L}_A \sqcup \mathsf{L}_B$.

If, in addition, the subproduct structure Π admits an orientation $O : \bar{E}_\Pi \to E_{AB}$, then the bottom graph L_A can be viewed as an oriented graph whose vertex set is A and the set of edges is $O(\bar{E}_{\Pi A})$. Similarly, the top graph L_B can be viewed as an oriented graph whose vertex set is B and the set of edges is $O(\bar{E}_{\Pi B})$. This turns L_{AB} into an oriented graph whose set of edges is $O(\bar{E}_\Pi)$.

2.6. **Taiko, the product graph.** Suppose a subproduct structure $\Pi = (A, B, P)$ is given whose subpartition P is either even or odd. The taiko *for* Π, or *the product graph* for Π, is the picture illustrating Π by placing A on the bottom, B on the top, drawing a vertical edge $(a, b) \in A \times B$, whenever there exists a 2-cell in P containing (a, b), and adding all the horizontal edges from the set $\bar{E}_\Pi$.

Furthermore, colors are used in taikos to indicate those horizontal edges that simultaneously occur in 2-cells of the partition P. More precisely, *a color of horizontal edges* is an equivalence class of unoriented horizontal edges, where the equivalence relation $\sim$ is the one generated by the relation $\sim'$: for two horizontal edges $\{a, a'\}$ and $\{b, b'\}$ we write $\{a, a'\} \sim' \{b, b'\}$ if there is a 2-cell in the partition P of the form $\{(a, b), (a', b')\}$ or $\{(a, b'), (a', b)\}$. With this definition, the 2-cells in P naturally inherit the same colors as their horizontal edges have, that is, there is a consistent coloring of 2-cells and horizontal edges. If there exists an orientation on $\bar{E}_\Pi$, we indicate it by placing an arrow on each unoriented horizontal edge as in Figures 1 and 2.

Figure 1 illustrates a particular *Bass unit* over a particular finite cyclic group (which clearly *does* have torsion). The size of the unit is $(m, n) = (7, 9)$ and the partition coming from that unit was further subdivided to make it odd. Figure 2 illustrates a particular product structure obtained by a computer search.

3. The topology of units and zero-divisors: origami and cell complexes.

For each product structure Π we define 2-dimensional cell complexes X_Π, Y_Π, $\bar{X}_\Pi$, $\bar{Y}_\Pi$ by an origami-like construction, by putting together several pieces of paper and folding them in a certain way.

3.1. **The complex X_Π.** To each product structure $\Pi = (A, B, P)$, where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, we associate a cell complex X_Π as follows. First consider the oriented graph with three vertices x_A, x_1, x_B, and two sets of oriented edges:

- the set of edges labeled by $a_1, \ldots, a_m$, each going from x_A to x_1,
- the set of edges labeled by $b_1, \ldots, b_n$, each going from x_1 to x_B.

The 2-dimensional cell complex X_Π is defined by attaching, for each 2-cell $\{(a_i, b_j), (a_{i'}, b_{j'})\}$, one square along the loop $a_i b_j b_{j'}^{-1} a_{i'}^{-1}$ in the graph (see Fig. 3). (In this way, the "2-cells" of the partition P exactly correspond to the "2-cells" in the cell complex X_Π.) G_Π will denote the fundamental group of X_Π with basepoint x_1. G_Π will be called *the universal group* associated with the product structure Π.

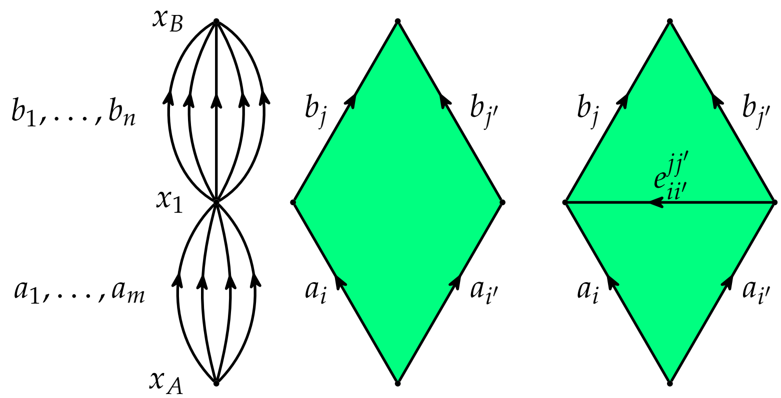


FIGURE 3. The complexes X_Π and Y_Π.

3.2. **The complex Y_Π.** The next complex, Y_Π, is obtained from X_Π by performing *2-foldings* as follows. First, for each 2-cell $\{(a_i, b_j), (a_{i'}, b_{j'})\}$ in the partition P draw an edge $e^{ii'}_{jj'}$ going from x_1 to x_1 within the corresponding 2-cell in X_Π, as illustrated in Fig 3. Such $e^{ii'}_{jj'}$ will be called *the middle edges*. The same edge $e^{ii'}_{jj'}$ considered with the opposite orientation will be denoted $(e^{ii'}_{jj'})^{-1}$. Next, whenever there are two 2-cells in the partition P of the form $\{(a_i, b_j), (a_{i'}, b_{j'})\}$ and $\{(a_i, b_{j''}), (a_{i'}, b_{j'''})\}$,

- identify the interiors of the two bottom triangles labeled $a_i^{-1}a_{i'}(e^{jj'}_{ii'})^{\varepsilon_1}$ and $a_i^{-1}a_{i'}(e^{j''j'''}_{ii'})^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$,
- identify the edges $e^{jj'}_{ii'}$ and $e^{j''j'''}_{ii'}$ if they have the same orientation in the sense that $\varepsilon_1 = \varepsilon_2$, and
- identify the edge $e^{jj'}_{ii'}$ with the edge opposite to $e^{j''j'''}_{ii'}$ if $e^{jj'}_{ii'}$ and $e^{j''j'''}_{ii'}$ and have opposite orientations, that is, if $\varepsilon_1 = -\varepsilon_2$.

Similarly, whenever there are two 2-cells in the partition P of the form $\{(a_i, b_j), (a_{i'}, b_{j'})\}$ and $\{(a_{i''}, b_j), (a_{i'''}, b_{j'})\}$, identify the two top triangles labeled $b_j b_{j'}^{-1}(e^{jj'}_{ii'})^{\varepsilon_1}$ and $b_j b_{j'}^{-1}(e^{jj'}_{i''i'''})^{\varepsilon_2}$. Keep performing such identifications for as long as possible. We let Y_Π be the 2-complex obtained at the end of this process.

It can be checked that the links of the vertices x_A and x_B in Y_Π are isomorphic to L_A and L_B, respectively. For this reason L_A and L_B can also be called *the bottom link* and *the top link*, respectively.

3.3. **The complexes $\bar{X}_\Pi$ and $\bar{Y}_\Pi$.** The complex $\bar{X}_\Pi$ is obtained from X_Π by identifying the three vertices x_A, x_1 and x_B into one vertex $\bar{x}_0$. The complex $\bar{Y}_\Pi$ is defined similarly: starting with the complex Y_Π, identify the vertices x_A, x_1 and x_B into one vertex $\bar{y}_0$. The fundamental group of $\bar{X}_\Pi$ will be denoted $\bar{G}_\Pi$ and will be called *the full universal group* of the product structure Π. In this quotient, the edges a_i and b_j become loops, so they represent elements of $\bar{G}_\Pi$, which we will also denote a_i and b_j, respectively.

$\bar{X}_\Pi$ can be equivalently described as the presentation complex of the presentation

$$\langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid a_i^{-1} a_{i'} b_{j'} b_j^{-1} \text{ for } \{(a_i, b_j), (a_{i'}, b_{j'})\} \in P \rangle.$$

By the van Kampen theorem, the group given by the above presentation is isomorphic to $\bar{G}_\Pi$. This presentation and its corresponding group seem to be folklore, having occurred in multiple places in the literature and on the internet; see, for example, [34], [21], [32].

3.4. **Associated units and zero-divisors over $\mathbb{Z}_2$.** Let $\Pi = (A, B, P)$ be a product structure in which the partition P is either odd or even. Let $\bar{a}_i$ and $\bar{b}_j$ denote the elements of $\bar{G}_\Pi \cong \pi_1(\bar{X})$ represented by the edge-loops labeled a_i and b_j, respectively.

Assume that mn is odd and P is odd. After relabeling we can assume that $\{(a_1, b_1)\}$ is the unique 1-cell in the partition P. Denote

$$a_\Pi := \bar{b}_1^{-1} \bar{a}_1^{-1} \sum_{i=1}^{m} \bar{a}_i = \sum_{i=1}^{m} \bar{b}_1^{-1} \bar{a}_1^{-1} \bar{a}_i \in \mathbb{Z}_2 \bar{G}_\Pi, \quad b_\Pi := \sum_{j=1}^{n} \bar{b}_j \in \mathbb{Z}_2 \bar{G}_\Pi.$$

For each 2-cell $\{(a_i, b_j), (a_{i'}, b_{j'})\}$ in the partition P, the edge-loops in $\bar{X}_\Pi$ labeled $a_i b_j$ and $a_{i'} b_{j'}$ are homotopic because the loop $a_i b_j b_{j'}^{-1} a_{i'}^{-1}$ bounds a (topological) 2-cell. This means that $\bar{a}_i \bar{b}_j = \bar{a}_{i'} \bar{b}_{j'}$ for each 2-cell $\{(a_i, b_j), (a_{i'}, b_{j'})\} \in P$, therefore, $\bar{a}_i \bar{b}_j + \bar{a}_{i'} \bar{b}_{j'} = 0 \in \mathbb{Z}_2 \bar{G}_\Pi$. Then

$$\begin{aligned} a_\Pi b_\Pi &= \bar{b}_1^{-1} \bar{a}_1^{-1} \sum_{i=1}^{m} \bar{a}_i \sum_{j=1}^{n} \bar{b}_j \\ &= \bar{b}_1^{-1} \bar{a}_1^{-1} \Big(\bar{a}_1 \bar{b}_1 + \sum_{\{(a_i, b_j), (a_{i'}, b_{j'})\} \text{ is a 2-cell in } P} (\bar{a}_i \bar{b}_j + \bar{a}_{i'} \bar{b}_{j'}) \Big) = \bar{b}_1^{-1} \bar{a}_1^{-1} \bar{a}_1 \bar{b}_1 = 1 \in \mathbb{Z}_2 \bar{G}_\Pi, \end{aligned}$$

i.e., a_Π and b_Π are units. They will be called *the units in $\mathbb{Z}_2 \bar{G}_\Pi$ associated with* Π.

If mn is even and P even, denote

$$a_\Pi := \sum_{i=1}^{m} \bar{a}_i \in \mathbb{Z}_2 \bar{G}_\Pi, \quad b_\Pi := \sum_{j=1}^{n} \bar{b}_j \in \mathbb{Z}_2 \bar{G}_\Pi.$$

Since P is an even partition,

$$a_\Pi b_\Pi = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{a}_i \bar{b}_j = \sum_{\{(a_i, b_j), (a_{i'}, b_{j'})\} \in P} (\bar{a}_i \bar{b}_j + \bar{a}_{i'} \bar{b}_{j'}) = 0 \in \mathbb{Z}_2 \bar{G}_\Pi,$$

i.e. a_Π and b_Π are zero-divisors. They will be called *the zero-divisors in $\mathbb{Z}_2 \bar{G}_\Pi$ associated with the product structure* Π.

Lemma 1. *If a product structure Π is nondegenerate and P is either odd or even, then $|supp(a_\Pi)| = m$ and $|supp(b_\Pi)| = n$.*

Proof. Assume that mn is odd and P is odd. If Π is nondegenerate, then the elements $\bar{b}_1^{-1}, \bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_2, \ldots, \bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_n \in \bar{G}_\Pi$ are pairwise distinct and the elements $\bar{b}_1, \ldots, \bar{b}_n \in \bar{G}_\Pi$ are pairwise distinct. This implies that the supports of a_Π and b_Π are the sets

$$supp(a_\Pi) = \{\bar{b}_1^{-1}, \bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_2, \ldots, \bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_n\}, \quad supp(b_\Pi) = \{\bar{b}_1, \ldots, \bar{b}_n\}$$

and $|supp(a_\Pi)| = m$, $|supp(b_\Pi)| = n$.

Now assume that mn is even and P is even. If Π is nondegenerate, then the elements $\bar{a}_1, \ldots, \bar{a}_m \in \bar{G}_\Pi$ are pairwise distinct and the elements $\bar{b}_1, \ldots, \bar{b}_n \in \bar{G}_\Pi$ are pairwise distinct. This implies that the supports of a_Π and b_Π are the sets

$$supp(a_\Pi) = \{\bar{a}_1, \ldots, \bar{a}_m\}, \quad supp(b_\Pi) = \{\bar{b}_1, \ldots, \bar{b}_n\}$$

and $|supp(a_\Pi)| = m$, $|supp(b_\Pi)| = n$. $\square$

3.5. **Associated units and zero-divisors over any ring with unity.** If $\Pi^\sigma = (A, B, P, \sigma)$ is a product structure with signature in which P is either odd or even, then we can produce associated units and zero-divisors in the algebra $R\bar{G}_\Pi$ over any ring R with unity, as follows.

First let $\Pi := (A, B, P)$ and, as before, let $\bar{G}_\Pi$ be the corresponding full universal group. If mn is odd and P is odd, we can assume that $\{(a_1, b_1)\}$ is the unique 1-cell in P and let

$$\begin{aligned}
a_\Pi &:= \bar{b}_1^{-1}\bar{a}_1^{-1}\sum_{i=1}^{m}\sigma(a_i)\bar{a}_i = \sum_{i=1}^{m}\sigma(a_i)(\bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_i) \in R\bar{G}_\Pi,\\
b_\Pi &:= \sum_{j=1}^{n}\sigma(b_j)\bar{b}_j \in R\bar{G}_\Pi,
\end{aligned}$$

then

$$\begin{aligned}
a_\Pi b_\Pi &= \bar{b}_1^{-1}\bar{a}_1^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}(\sigma(a_i)\sigma(b_j))(\bar{a}_i\bar{b}_j)\\
&= \bar{b}_1^{-1}\bar{a}_1^{-1}\Big(\bar{a}_1\bar{b}_1 + \sum_{\{(a_i,b_j),(a_{i'},b_{j'})\}\text{ is a 2-cell in }P}((\sigma(a_i)\sigma(b_j))(\bar{a}_i\bar{b}_j) + (\sigma(a_{i'})\sigma(b_{j'}))\bar{a}_{i'}\bar{b}_{j'})\Big)\\
&= \bar{b}_1^{-1}\bar{a}_1^{-1}\Big(\bar{a}_1\bar{b}_1 + \sum_{\{(a_i,b_j),(a_{i'},b_{j'})\}\text{ is a 2-cell in }P}(\sigma(a_i)\sigma(b_j))(\bar{a}_i\bar{b}_j - \bar{a}_{i'}\bar{b}_{j'})\Big)\\
&= \bar{b}_1^{-1}\bar{a}_1^{-1}\bar{a}_1\bar{b}_1 = 1 \in R\bar{G}_\Pi,
\end{aligned}$$

i.e., a_Π and b_Π are units in $R\bar{G}_\Pi$, which we will call *the units over R associated with* Π^σ.

If mn is even and P is even, let

$$a_\Pi := \sum_{i=1}^{m}\sigma(a_i)\,\bar{a}_i \in R\bar{G}_\Pi, \quad b_\Pi := \sum_{i=1}^{m}\sigma(b_j)\,\bar{b}_j \in R\bar{G}_\Pi,$$

then

$$\begin{aligned} a_\Pi b_\Pi &= \sum_{i=1}^{m}\sum_{j=1}^{n}(\sigma(a_i)\sigma(b_j))(\bar{a}_i\bar{b}_j) \\ &= \sum_{\{(a_i,b_j),(a_{i'},b_{j'})\}\in P} ((\sigma(a_i)\sigma(b_j))\bar{a}_i\bar{b}_j + (\sigma(a_{i'})\sigma(b_{j'}))\bar{a}_{i'}\bar{b}_{j'}) \\ &= \sum_{\{(a_i,b_j),(a_{i'},b_{j'})\}\in P} (\sigma(a_i)\sigma(b_j))(\bar{a}_i\bar{b}_j - \bar{a}_{i'}\bar{b}_{j'}) = 0 \in R\bar{G}_\Pi, \end{aligned}$$

i.e., a_Π and b_Π are zero-divisors in $R\bar{G}_\Pi$, which we will call *the zero-divisors over R associated with* Π^σ. The proof of the following lemma is similar to the proof of Lemma 1.

Lemma 2. *If a product structure with signature Π^σ is nondegenerate, then $|supp(a_\Pi)| = m$ and $|supp(b_\Pi)| = n$.*

3.6. **The fundamental groups of X_Π, Y_Π, $\bar{X}_\Pi$, $\bar{Y}_\Pi$.** We have the following diagrams for the four complexes and their fundamental groups

$$\text{(1)} \qquad \begin{array}{ccc} X_\Pi & \xrightarrow{q_\Pi} & Y_\Pi \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\ \bar{X}_\Pi & \xrightarrow{\bar{q}_\Pi} & \bar{Y}_\Pi \end{array} \qquad\qquad \begin{array}{ccc} G_\Pi = \pi_1(X_\Pi) & \xrightarrow{q_{\Pi*}} & \pi_1(Y_\Pi) \\ \downarrow{\scriptstyle q_{X*}} & & \downarrow{\scriptstyle q_{Y*}} \\ \bar{G}_\Pi = \pi_1(\bar{X}_\Pi) & \xrightarrow{\bar{q}_{\Pi*}} & \pi_1(\bar{Y}_\Pi) \end{array}$$

where q_X, q_Y, q_Π, $\bar{q}_\Pi$ are the canonical quotient maps. The maps q_X, q_Y, q_Π are given by the definitions of the corresponding complexes, and $\bar{q}_\Pi$ is defined by performing the 2-foldings on the complex $\bar{X}_\Pi$ in parallel to the 2-foldings on X_Π in the definition of Y_Π. The homomorphisms q_{X*}, q_{Y*}, $q_{\Pi*}$, $\bar{q}_{\Pi*}$ are the corresponding induced homomorphisms.

Lemma 3. *The following properties hold for each product structure Π.*

(a) *The diagrams in (1) are commutative.*

(b) *The full universal group $\bar{G}_\Pi = \pi_1(\bar{X}_\Pi)$ is isomorphic to the free product $G_\Pi * F_2$, where F_2 is the free group of rank 2. With the identification $\bar{G}_\Pi \cong G_\Pi * F_2$, the homomorphism $q_{X*} : G_\Pi \to \bar{G}_\Pi$ induced by the quotient map $q_X : X_\Pi \twoheadrightarrow \bar{X}_\Pi$ is the same as the standard inclusion $G_\Pi \hookrightarrow G_\Pi * F_2$ onto the factor G_Π. In particular, q_{X*} is injective.*

(c) *The group $\pi_1(\bar{Y}_\Pi)$ is isomorphic to the free product $\pi_1(Y_\Pi) * F_2$. With the identification $\pi_1(\bar{Y}_\Pi) \cong \pi_1(Y_\Pi) * F_2$, the homomorphism $q_{Y*} : \pi_1(Y_\Pi) \to \pi_1(\bar{Y}_\Pi)$ induced by the quotient map $q_Y : Y_\Pi \twoheadrightarrow \bar{Y}_\Pi$ is the same as the standard inclusion $\pi_1(Y_\Pi) \hookrightarrow \pi_1(Y_\Pi) * F_2$ onto the factor $\pi_1(Y_\Pi)$. In particular, q_{Y*} is injective.*

Proof. (a) The left diagram commutes because the 2-foldings in X_Π and in $\bar{X}_\Pi$ do not affect their vertices. The right diagram commutes because it is induced by the left one.

(b) $\bar{X}_\Pi$ is obtained from X_Π by identifying three vertices x_A, x_1 and x_B into one. This operation can be split into two steps: first attach an two edges connecting x_A to x_1 and x_1 to x_B, respectively, then collapse each edge to a point. The second step is a homotopy equivalence, and, since X_Π is path-connected, the result of the first step is homotopy equivalent to the wedge sum of X_Π with two circles, $X_\Pi \vee S^1 \vee S^1$. Then by the van Kampen theorem,

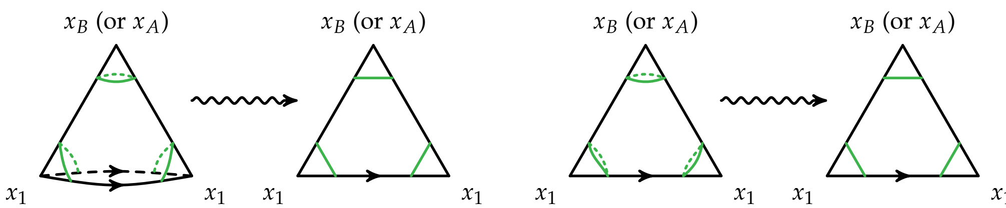
$$\bar{G}_\Pi = \pi_1(\bar{X}_\Pi) \cong \pi_1(X_\Pi \vee S^1 \vee S^1) \cong \pi_1(X_\Pi) * F_2 = G_\Pi * F_2.$$

The proof of (c) is similar. □

3.7. **When** $\pi_1(Y_\Pi) \cong \pi_1(X_\Pi)$ **and** $\pi_1(\bar{Y}_\Pi) \cong \pi_1(\bar{X}_\Pi)$**.** In general, the 2-foldings used to obtain Y_Π and $\bar{Y}_\Pi$ do not necessarily preserve the homotopy type, but we now show that they do preserve the fundamental group *if* Π is orientable as defined in section 2.4.

Lemma 4. *If* Π *is orientable, then* $\pi_1(Y_\Pi) \cong \pi_1(X_\Pi) = G_\Pi$ *and* $\pi_1(\bar{Y}_\Pi) \cong \pi_1(\bar{X}_\Pi) = \bar{G}_\Pi$*, and the homomorphisms* $q_{\Pi *}$ *and* $\bar{q}_{\Pi *}$ *in diagram (1) are isomorphisms.*

Proof. Y_Π is obtained from X_Π by performing 2-foldings. Since Π is orientable, there is a consistent orientation of the middle edges, hence at each step a 2-folding is of one of the following two types:



The edges drawn on the bottom represent some middle edges (or their images after gluing). The two vertices of each middle edge actually coincide in X_Π; we draw them distinct in the picture to illustrate the 2-foldings clearly.

The 2-folding of type 1, pictured on the left, is a homotopy equivalence, so it does not change the fundamental group. The 2-folding of type 2, pictured on the right, can be equivalently described as follows. The two triangles share all their three sides, so that their union V is homeomorphic to the 2-sphere. First attach a closed 3-disk D^3 to V by identifying its boundary with V by a homeomorphism. Attaching a 3-ball is not a homotopy equivalence, but it preserves the fundamental group, by the Van Kampen theorem, because both D^3 and the image of its boundary S^2 are simply connected. Then collapse the attached 3-disk so that the two triangles are identified. This operation is a homotopy equivalence, so we have isomorphisms $\pi_1(Y_\Pi) \cong \pi_1(X_\Pi) = G_\Pi$.

To prove the existence of an isomorphism $\pi_1(\bar{Y}_\Pi) \cong \pi_1(\bar{X}_\Pi) = \bar{G}_\Pi$, perform the same 2-foldings on $\bar{X}_\Pi$ to obtain $\bar{Y}_\Pi$. The same argument applies. □

If Π is not orientable, after performing 2-foldings some middle edge e in the above figure will eventually be identified with its inverse; this amounts to folding e in half. Since each such edge e is actually a loop, it might represent a nontrivial element in the fundamental group, and folding it in half makes it nullhomotopic in the quotient, therefore potentially changing the fundamental group.

3.8. **Nondegeneracy and nontriviality.** For any product substructure Π, the elements in $A = \{a_1, \ldots, a_m\}$ and in $B = \{b_1, \ldots, b_n\}$ one-to-one correspond to the edges in X_Π, also labeled $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$. These edges become loops in the complex $\bar{X}_\Pi$, so they represent some elements of the full universal group $\bar{G}_\Pi = \pi_1(\bar{X}_\Pi)$, which we denote $\bar{a}_1, \ldots, \bar{a}_m$ and $\bar{b}_1, \ldots, \bar{b}_n$, respectively. A product structure or substructure Π is called *nondegenerate* if $\bar{a}_i \neq \bar{a}_{i'}$ for $i \neq i'$ and $\bar{b}_j \neq \bar{b}_{j'}$ for $j \neq j'$. We will use the same definition of nondegeneracy for product structures with signature. The following equivalent description of nondegeneracy follows from Lemma 3.

Lemma 5. *A product structure or substructure Π is nondegenerate if and only if*

- *for each pair $(a_i, a_{i'})$ with $i \neq i'$, the loop in X_Π at x_1 with label $a_i^{-1} a_{i'}$ represents a non-trivial element in the universal group $G_\Pi = \pi_1(X_\Pi)$, and*
- *for each pair $(b_j, b_{j'})$ with $j \neq j'$, the loop in X_Π at x_1 with label $b_j b_{j'}^{-1}$ represents a non-trivial element in G_Π.*

The support of an element $a = \sum_{g \in G} r_g\, g$ in a group ring RG is the set

$$supp(a) = \{g \mid r_g \neq 0\}.$$

A unit a in a group ring RG is *nontrivial* if a is not an R-multiple of a single element in G, that is, if $|supp(a)| \geq 2$. A zero-divisor a in a group ring RG is *nontrivial* if $a \neq 0$, that is, if $|supp(a)| \geq 1$. Lemmas 1 and 2 imply the following.

Lemma 6. *If a product structure Π is nondegenerate, Π is of size (m, n), $m \geq 2$ and $n \geq 2$, then the associated units or zero-divisors $a_\Pi, b_\Pi \in \mathbb{Z}_2\bar{G}_\Pi$ are nontrivial. If a product structure with signature Π^σ is nondegenerate, Π^σ is of size (m, n), $m \geq 2$ and $n \geq 2$, then the associated units or zero-divisors $a_\Pi, b_\Pi \in R\bar{G}_\Pi$ are nontrivial.*

The fact that Fig. 1 comes from a known nontrivial unit implies that it represents a nondegenerate product structure. For Fig. 2 nondegeneracy is not immediately apparent, and generally there is no known way to verify nondegeneracy.

Generally, deciding whether two elements in a group G given by a presentation are equal is equivalent to the word problem introduced by Max Dehn in 1911 [20]. Novikov [29], [30], [31] and Boone [12], [13], [14] independently proved that the word problem is unsolvable, in general, for finitely presented groups. Since there are many types of product structures, it is reasonable to expect that the question whether a given product structure Π is nondegenerate should be hard or impossible to decide algorithmically in general. This is where geometry helps: we will show that a product structure Π is nondegenerate *if* the complex Y_Π admits a metric of negative curvature. Further, we will list specific combinatorial conditions guaranteeing that Y_Π admits a metric of negative curvature. This makes it possible to utilize computer search to look for nontrivial units and zero divisors.

4. Geometry of metric spaces

One benefit of working with riemannian manifolds is that one can define and use a notion of sectional curvature. There is an extensive bibliography of articles and books generalizing this notion to more general metric spaces, particularly, to simplicial complexes and cell complexes. In this section we summarize known results that allow, under certain assumptions, to put a nice metric structure on a given cell complex.

4.1. **Length spaces, geodesic spaces.** *A length space* is a metric space X in which the distance between every pair of points $x, y \in X$ is equal to the infimum of the lengths of rectifiable curves joining them. *A geodesic*, or *a geodesic path*, in a metric space X is an isometric embedding of an interval into X. *A geodesic metric space* is a metric space in which each pair of points can be connected by a geodesic. *The length metric* on a metric space is defined as the infimum of the lengths of rectifiable curves; we refer to [16, I.3.2, I3.3, pp. 32-33] for details.

4.2. **Spaces of curvature $\leq \kappa$ and $CAT(\kappa)$-spaces.** In 1948 Busemann [18] introduced a notion of a nonpositively curved space using upper bounds on the middle lines of geodesic triangles. In 1951 Aleksandrov [2, §1.3, p. 8 and §4, p. 19] gave a general definition of spaces of curvature bounded above. Aleksandrov's earlier works also discussed notions of curvature for metric spaces, in particular the notion of curvature bounded below; see [1], [6].

Given a real number κ, a $CAT(\kappa)$*-space* is a geodesic metric space in which each geodesic triangle is at least as thin as its comparison triangle in the standard (simply connected) space of curvature κ; see [16, II.1.1, pp. 158-159] for precise definitions. A comparison triangle is a triangle with the same side lengths as those of the original triangle. For $\kappa = 0$, the standard space is the euclidean plane, for $\kappa = -1$ it is the hyperbolic plane, and for $\kappa = 1$ it is the unit 2-dimensional sphere. This in particular implies that if $\kappa \leq 0$, for each pair of points x and y in a $CAT(\kappa)$-space, the geodesic joining x to y is unique.

A metric space is said to be of *curvature* $\leq \kappa$ if it is locally a $CAT(\kappa)$-space; see [16, p. 159]. A space of curvature ≤ 0 is also called *a nonpositively curved space*.

4.3. **The Cartan-Hadamard theorem.** This theorem relates spaces of curvature $\leq \kappa$ to $CAT(\kappa)$-spaces. The original statement of the theorem was proved by Hadamard [24] in the case of surfaces and by Cartan [19] for arbitrary riemannian manifolds of nonpositive curvature. This is an example of a local-to-global result, i.e., deducing properties of the universal covering from the local structure of a space. The following theorem is a generalization of the original Cartan-Hadamard theorem from manifolds to metric spaces. It is a variation of theorem stated by Gromov [23, p.119], a detailed proof of which was given by Ballmann in the locally compact case [7], Alexander-Bishop [5] proved this result under the additional assumption that $\tilde{X}$ is a geodesic metric space.

Theorem 7 (The Cartan-Hadamard theorem [24], [19], [23, p.119], [7], [5], [16, II.4.1, p.193])**.** *Let X be a complete connected metric space.*

(1) *If the metric on X is locally convex, then the induced length metric on the universal cover $\tilde{X}$ is (globally) convex. (In particular, there is a unique geodesic segment joining each pair of points in $\tilde{X}$ and geodesic segments vary continuously with their endpoints.)*
(2) *If X is of curvature $\leq \kappa$, where $\kappa \leq 0$, then $\tilde{X}$ (with the induced length metric) is a $CAT(\kappa)$-space.*

4.4. **Fixed points.** Certain notions of *center* for a given subset $Y \subseteq X$ were considered by Cartan [19] for any simply connected manifold M of nonpositive curvature. He used this notion to prove the existence of a fixed point for the action of any compact group of isometries of M. Bruhat and Tits [17] proved a similar theorem for group actions on euclidean buildings.

We now quote similar theorems in the more general case of $CAT(\kappa)$ spaces. Define *the radius* of a subset Y in a metric space X to be the infimum of the positive numbers r such that $Y \subseteq B(x, r)$ for some $x \in X$.

Theorem 8 ([16, II.2.7, p. 179]). *Let X be a complete $CAT(\kappa)$ space and $Y \subseteq X$ be a bounded subset. If $\kappa > 0$, assume additionally that the radius of Y is $< \pi/(2\sqrt{\kappa})$. Then there exists a unique point $c_Y \in X$, called the center of Y, such that $Y \subseteq \bar{B}(c_Y, r_Y)$.*

The following general theorem is sometimes called the Cartan fixed-point theorem or the Bruhat-Tits fixed-point theorem.

Theorem 9 ([16, II.2.8, p. 179]). *If X is a complete $CAT(0)$ space and Γ is a finite group of isometries of X or, more generally, a group of isometries with a bounded orbit, then the fixed-point set of Γ is a non-empty convex subset of X.*

4.5. **Nonpositive curvature implies "torsion-free".** For any complete, path-connected, nonpositively curved metric space X, consider the action of its fundamental group $\pi_1(X)$ on the universal covering $\tilde{X}$ (see, for example, [25, Ch. 1, Prop. 1.39, p. 71]). The action is defined by representing each $g \in \pi_1(\tilde{X})$ by a loop f, lifting f to a path $\tilde{f}$ in $\tilde{X}$, and defining the unique deck transformation that sends $\tilde{f}(0)$ to $\tilde{f}(1)$. If an element $g \in \pi_1(X)$ fixes a point in $\tilde{X}$, then $\tilde{f}$ is a loop, hence it is nullhomotopic, then so is f, so g is trivial. In other words, the $\pi_1(X)$-action on $\tilde{X}$ is free. By the Cartan-Hadamard theorem (Theorem 7), $\tilde{X}$ is a $CAT(0)$ space. If there were a nontrivial element $g \in \pi_1(X)$ of finite order, then by Theorem 9, g must have a fixed point in $\tilde{X}$, which is a contradiction. This proves the following.

Theorem 10 ([16, II.4.13, p. 201]). *Let X be a complete, path-connected, nonpositively curved metric space. Then the group $\pi_1(X)$ is torsion-free.*

4.6. **The Berestovskii theorem.** The notion of a κ-cone over a metric space is due to Berestovskii. (See [10, Def. 1], [9, Def. 1], [4, II.4.3, pp. 14–15], [3, II.4.3, p. 17], [11, 3.2, pp. 205–206], [8, 3.2, pp. 180–181], [16, Def. 5.6, p. 59]. This last name is also spelled as "Berestovskiĭ" and ""Berestovskij" in the literature.)

The main idea to define a metric on the cone is to use the three laws of cosines: one for curvature $\kappa = 0$ (in the euclidean plane), one for constant negative curvature $\kappa < 0$ (in a rescaled hyperbolic plane), and one for constant positive curvature $\kappa > 0$ (in a rescaled

2-sphere). Specifically,

$$\begin{aligned}
&\text{for } \kappa = 0: \quad c^2 = a^2 + b^2 - 2ab\cos\gamma,\\
&\text{for } \kappa < 0: \quad \cosh(\sqrt{-\kappa}c) = \cosh(\sqrt{-\kappa}a)\cosh(\sqrt{-\kappa}b)\\
&\qquad - \sinh(\sqrt{-\kappa}a)\sinh(\sqrt{-\kappa}b)\cos\gamma,\\
&\text{for } \kappa > 0: \quad \cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a)\cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a)\sin(\sqrt{\kappa}b)\cos\gamma.
\end{aligned}$$

Let (Y, d_Y) be a metric space and κ be a real number. The *κ-cone* over Y is the metric space $(C_\kappa Y, d)$ defined as follows. First denote

$$I_\kappa := \begin{cases} [0,\infty) & \text{if } \kappa \le 0,\\ [0, \pi/(2\sqrt{\kappa})] & \text{if } \kappa > 0.\end{cases}$$

Let $C_\kappa Y$ be the quotient of the set $I_\kappa \times Y$ by the equivalence relation

$$(t,y) \sim (t',y') \quad \Leftrightarrow \quad t = t' = 0 \text{ or } (t = t' \text{ and } y = y').$$

Denote the equivalence class of (t, y), i.e., the point in the cone corresponding to (t, y) by ty. Depending on κ, the distance between two points ty and $t'y'$ in the cone $C_\kappa Y$ is defined by solving for c in the corresponding law of cosines for $a := t$, $b := t'$, and $\gamma := \min\{d_Y(y, y'), \pi\}$. Specifically,

$$\begin{aligned}
&\text{for } \kappa = 0: \quad d(ty, t'y') := \sqrt{t^2 + t'^2 - 2tt'\cos(\min\{\pi, d_Y(y,y')\})},\\
&\text{for } \kappa < 0: \quad d(ty, t'y') := \cosh^{-1}\Big(\Big[\cosh(\sqrt{-\kappa}a)\cosh(\sqrt{-\kappa}b)\\
&\qquad - \sinh(\sqrt{-\kappa}a)\sinh(\sqrt{-\kappa}b)\cos(\min\{\pi, d_Y(y,y')\})\Big]/\sqrt{-\kappa}\Big),\\
&\text{for } \kappa > 0: \quad d(ty, t'y') := \arccos\Big(\Big[\cos(\sqrt{\kappa}a)\cos(\sqrt{\kappa}b)\\
&\qquad + \sin(\sqrt{\kappa}a)\sin(\sqrt{\kappa}b)\cos(\min\{\pi, d_Y(y,y')\})\Big]/\sqrt{\kappa}\Big).
\end{aligned}$$

The proof of the following theorem due to Berestovskii can be found in [10], [9], [4], [3], [16, II.3.14, pp. 188-190].

Theorem 11 (Berestovskii)**.** *Let Y be a metric space. The κ-cone $X = C_\kappa Y$ over Y is a* CAT(κ)*-space if and only if Y is a* CAT(1)*-space.*

Now we prove the lemma that will later be helpful for proving nondegeneracy of product structures. For each point $y \in Y$ there is a canonical interval in the cone $C_\kappa Y$ that "connects y to the cone point". Specifically, it is the image of the interval I_κ under *the canonical embedding* $\iota_y : I_\kappa \hookrightarrow C_\kappa Y$ given by $\iota_y(t) := ty$.

Lemma 12. *Let Y be a metric space. For every $y \in Y$, the canonical embedding $\iota_y : I_\kappa \hookrightarrow C_\kappa Y$ is an isometric embedding.*

Proof. In the case $\kappa = 0$,

$$d(\iota_y(t), \iota_y(t')) = d(ty, t'y) = \sqrt{t^2 + t'^2 - 2tt' \cos(\min\{\pi, d_Y(y, y)\})}$$
$$= \sqrt{t^2 + t'^2 - 2tt'} = |t - t'|.$$

Since we will only use this lemma in the case $\kappa = 0$, the cases $\kappa < 0$ and $\kappa > 0$ are left as an exercise. □

4.7. **Local and global isometric embeddings.** The following theorem is another example of a local-to-global property.

Theorem 13 ([16, Prop. 4.14, p. 201])**.** *Let X and Y be complete connected metric spaces. Suppose that X is non-positively curved and that Y is is locally a length space. If there is a map $f : X \to Y$ that is locally an isometric embedding, then Y is non-positively curved and:*

(1) *For every $y_0 \in Y$, the homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(X, f(x_0))$ induced by f is injective.*
(2) *Consider the universal coverings $\tilde{X}$ and $\tilde{Y}$ with their induced length metrics. Every continuous lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of f is an isometric embedding.*

5. Geometry of cell complexes

5.1. M_κ**-simplicial and** M_κ**-polyhedral complexes.** For $\kappa \in (0, \infty)$, M_κ denotes the standard (simply connected) n-dimensional space of constant curvature κ. For example, M_0^n is the n-dimensional euclidean space, M_{-1} is the n-dimensional hyperbolic space, and M_1^n is the n-dimensional sphere of radius 1. *An n-dimensional M_κ-simplex* is the convex hull of $n + 1$ points in M_κ in general position; in the case $\kappa = 1$ we additionally require that those points lie in some ball of radius $< \pi/2$ (i.e., in some open hemisphere) in the sphere M_1^n.

An M_κ-simplicial complex is built out of a family of M_κ-simplices by identifying certain faces of those simplices by isometries; we refer to [16, I.7.2,p.98] for the precise definition. It is required that the map of each simplex to the quotient space is injective. A *convex M_κ-cell* is the convex hull of finitely many points in the standard space M_κ^n. An *M_κ-polyhedral complex* is built similarly from a family of convex M_κ-cells by gluing certain faces of those cells; we refer to [16, I.7.37, p.114] for the precise definition. The map from each cell C to the quotient is required to be injective only *on the interior* of the face; it is allowed, for example, to glue together some faces of C.

Let K be an M_κ-polyhedral complex. Each cell C in K comes with the metric d_C induced from its inclusion into M_κ^n. *The intrinsic pseudometric d* on K is defined using piecewise geodesic paths: for $x, y \in K$,

$$d(x, y) := \inf_c l(c), \tag{2}$$

where $c : [a, b] \to K$ is a path from x to y that is subdivided as a concatenation of geodesic pieces c_i so that the image of each c_i lies in some cell C_i of K, $l(c_i)$ is the length of c_i with respect to the original metric on C_i, and $l(c) := \sum_{i=1}^k l(c_i)$. Equivalently, instead of piecewise geodesic paths one can use strings of points. Also, *the quotient pseudometric* can be defined using particular finite sequences of points. See [16, I.7.4 and I.5.19, p.65], [16, I.7.4 and I.7.5, p.99] and [16, I.7.38, p.114] for the equivalence of all these definitions.

One can easily switch between simplicial and polyhedral M_κ-complexes because each simplicial M_κ-complex is a polyhedral M_κ-complex, and each polyhedral M_κ-complex can be subdivided to become a simplicial M_κ-complex with the same metric; see [16, Proposition I.7.49, p.118].

For an M_κ-polyhedral complex K, Shapes(K) denotes the set of all isometry classes of the M_κ-cells in K.

Theorem 14 (Bridson [15], [16, I.7.19, p.105 and I.7.50, p.118])**.** *Let K be a connected M_κ-simplicial or M_κ-polyhedral complex. If Shapes(K) is finite, then K is a complete geodesic metric space.*

5.2. **Links.** Given a vertex x in K, *the link* of x, denoted $\mathrm{Lk}(x,K)$, is the set of all directions at x in K; see [16, I.7.15, p.103]. Informally, one can think of the directions at x as all the unit tangent vectors at x that "point inside" the (closed) cells containing x. For each n-cell C in K containing x, we can view $\mathrm{Lk}(x,C)$ as a $(n-1)$-cell in $\mathrm{Lk}(x,K)$. Furthermore, using angles between directions, $\mathrm{Lk}(x,C)$ can be identified with an $(n-1)$-simplex lying inside the $(n-1)$-dimensional sphere, that is, of M_1^{n-1}. In this way, the structure of M_κ-polyhedral complex on K induces a structure of an M_1-polyhedral complex on each link $\mathrm{Lk}(x,K)$.

5.3. **Local geometry of M_κ-polyhedral complexes.** For an M_κ-simplicial complex X and $x \in K$, define

$$\varepsilon(x) := \inf\{\varepsilon(x,S) \mid S \subseteq K \text{ is a simplex containing } x\},$$

where

$$\varepsilon(x,S) := \inf\{d_S(x,T) \mid T \text{ is a face of } S \text{ and } x \notin T\}.$$

Here d_S is the original metric on the metric simplex S, the one coming from its inclusion into M_κ. More generally, if K is an M_κ-polyhedral complex and $x \in K$, $\varepsilon(x)$ can be defined similarly as the distance from x to the closure of its star $st(x)$ minus $st(x)$, where the distance is measured in terms of the original metric on each cell in K; see [16, I.7.38, p. 114] for details. If the set $Shapes(K)$ is finite, then $\varepsilon(x) > 0$.

Theorem 15 ([16, I.7.16, pp. 103-105])**.** *Let K be an M_κ-simplicial complex and let $x \in K$. If $\varepsilon(x) > 0$, then $B(x,\varepsilon(x)/2)$ is naturally isometric to the open ball of radius $\varepsilon(x)/2$ about the cone point in $C_\kappa(\mathrm{Lk}(x,K))$.*

Lemma 16 ([16, I.7.56, p. 120])**.** *Let K be an M_κ-polyhedral complex with $Shapes(K)$ finite. If the points x and y lie in in the same open cell in K, then for sufficiently small $\varepsilon > 0$ there exists an isometry from $B(x,\varepsilon)$ to $B(y,\varepsilon)$ that restricts to an isometry from $B(x,\varepsilon) \cap C$ to $B(y,\varepsilon) \cap C$ for every closed cell C containing x.*

Lemma 17. *Let K be an M_κ-polyhedral complex with $Shapes(K)$ finite. For any edge (=1-cell) e in K, the canonical map $c_e : e \to K$ is locally an isometric embedding.*

Proof. The 1-cell e is topologically a closed interval, though the canonical map in general might identify its endpoints, so c_e it is not necessarily injective. Let v be a vertex in K corresponding to an endpoint of e. By Theorem 15, some neighborhood V of v in K is isometric to some neighborhood of the cone point in the κ-cone $C_\kappa(\mathrm{Lk}(v, K))$. Furthermore, following the proof of that theorem one can see that this isometry is consistent with the cellular structures of V and of the link $\mathrm{Lk}(v, K)$ (both induced from K). In particular, each vertex in the link $\mathrm{Lk}(v, K)$ corresponds to an edge in the cone $C_\kappa(\mathrm{Lk}(v, K))$, and that edge is mapped to an edge in V isometrically. By Lemma 12 the canonical embedding $I_\kappa \hookrightarrow C_\kappa(\mathrm{Lk}(v, K))$ is an isometry. This implies that if a is an endpoint of the edge e, then there exists a neighborhood of a in e such that the restriction of the canonical map c_e to this neighborhood is an isometric embedding.

Now take a to be any point in the interior of e in K, and let b be a point in the interior of e in K that is mapped into V by c_e. Since some neighborhood of b in e is mapped isometrically into K, then by Lemma 16 some neighborhood of a in e is also mapped isometrically. This proves that c_e is locally an isometric embedding. □

5.4. **Complexes of curvature $\leq \kappa$.** The following link condition was introduced by Gromov [23]. An M_κ-polyhedral complex K with $Shapes(X)$ finite is said to satisfy *the link condition* if for every vertex $v \in K$ the link complex $\mathrm{Lk}(v, K)$ is a $CAT(1)$ space. The following result was first stated by Gromov [23, p. 120], proved by Ballmann [7] in the locally compact case and by Bridson [15] in the general case.

Theorem 18 ([16, section II.5.5, p. 207])**.** *Let X be an M_κ-polyhedral cell complex such that $Shapes(X)$ is finite. The curvature of X is bounded above by κ if and only if X satisfies the link condition.*

An injective loop in a graph $\mathcal{G}$ can be defined as an injective continuous function from a circle to $\mathcal{G}$. *A simple closed curve* in $\mathcal{G}$ can be defined as a continuous function $[0, 1] \to \mathcal{G}$, which is injective except the two endpoints 0 and 1 are mapped to the same point in $\mathcal{G}$. These two notions uniquely determine each other, and can be used interchangeably in what follows. It is not hard to see that that a metric graph $\mathcal{G}$ is a $CAT(1)$-space if an only if it has metric girth $\geq 2\pi$, i.e., every injective loop in $\mathcal{G}$ is of length $\geq 2\pi$. This implies the following lemma.

Lemma 19 ([16, section II.5.6, p. 207])**.** *A 2-dimensional M_κ-complex K satisfies the link condition if and only if for each vertex v in K every injective loop in* $\mathrm{Lk}(v, K)$ *has length at least* 2π*.*

The proof of the following theorem is obtained by combining Theorem 18 and Lemma 19.

Theorem 20. *Let X be a 2-dimensional piecewise euclidean cell complex. X is non-positively curved if and only if at each vertex $v \in X$, each injective loop in the link of v is of length at least 2π.*

6. The geometry of units and zero-divisors

6.1. **The middle link.** *The middle link* of a product structure or substructure Π, denoted L_1, is the link of the middle vertex x_1 in the complex Y_Π.

A pattern in a taiko is an unordered pair of colors of horizontal edges together with their orientations, that occur incident at a common vertex v in $\mathsf{L}_{AB} = \mathsf{L}_A \sqcup \mathsf{L}_B$. The situation when the same pattern occurs at least twice at different vertices in L_{AB} will be called *a repeating pattern*.

A simplified version of the middle link, denoted L_1^{sim}, is the graph whose vertices are the pairs $(horizontal\text{-}edge\text{-}color, direction)$ and the edges are the patterns occurring in L_A and L_B. The two graphs in Figure 4 illustrate the simplified middle links of the two taikos in Figures 1 and 2, respectively.

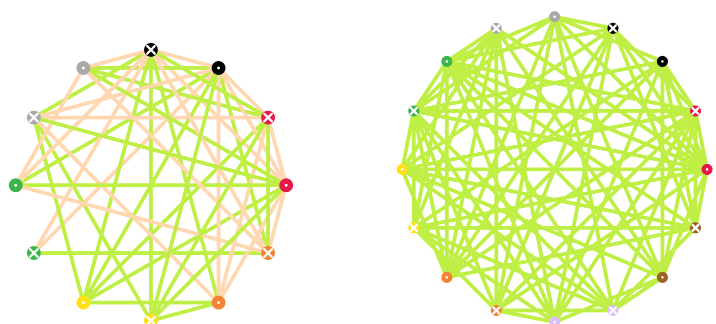
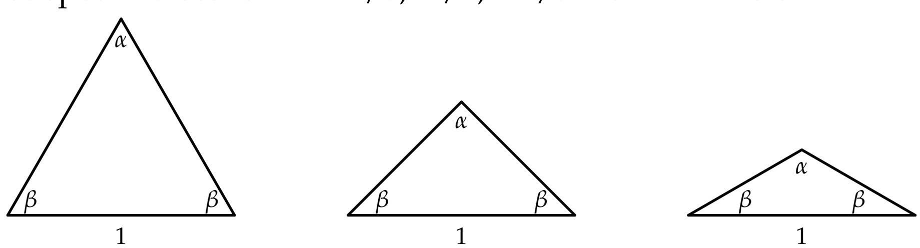


FIGURE 4. *Product structure example 1:* $(m, n) = (7, 9)$. The simplified middle link L_1^{sim} has 12 vertices and 44 edges. Some patterns repeat. *Product structure example 2:* $(m, n) = (8, 5)$. The simplified middle link L_1^{sim} has 16 vertices and 78 edges. There are no repeating patterns. Lime-colored edges represent patterns occurring once. Apricot-colored edges represent repeating patterns.

6.2. **Metric structures on Y_Π.** For each $\alpha \in (0, \pi)$, let Δ_α be the isosceles triangle in the euclidean plane with base of length 1 and the angles (α, β, β), then necessarily $\beta = (\pi - \alpha)/2$. The three special cases for $\alpha = \pi/3$, $\pi/2$, $2\pi/3$ are drawn below.



Now we turn the cell complex Y_Π defined in section 3.2 into a metric space: first identify each of its 2-cells with the isosceles euclidean triangle Δ_α so that the corners with angle α are either on the top or on the bottom. Put the intrinsic pseudometric on Y_Π as defined in (2). Y_Π is not a simplicial complex because some vertices of its triangles are identified, but it is an M_0-polyhedral complex. Furthermore, it becomes an M_0-simplicial complex after taking the first barycentric subdivision.

6.3. **Multiplicity.** The angles α and β are uniquely determined by each other. Specifically, for $\alpha \in (0, \pi)$, $\beta = \bar{\beta}(\alpha) := (\pi - \alpha)/2$, and the function $\bar{\beta}$ is a decreasing bijection $(0, \pi) \to$

$(0, \pi/2)$. We will be interested in an explicit description of the set

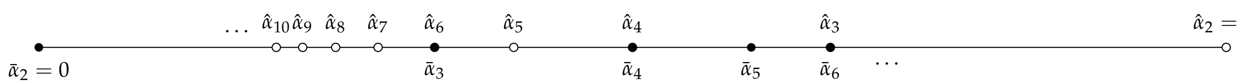
$$Q := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid \exists \alpha \in (0, \pi) \quad \alpha i \geq 2\pi \text{ and } 2\bar{\beta}(\alpha)\, j \geq 2\pi\}.$$

To address this question, for $\alpha \in (0, \pi)$, define *the α-multiplicity* function $\hat{\mu} : (0, \pi) \to \mathbb{Z}$ and *the β-multiplicity* function $\bar{\mu} : (0, \pi) \to \mathbb{Z}$ by the formulas

$$\hat{\mu}(\alpha) := \min\{k \in \mathbb{Z} \mid \alpha k \geq 2\pi\} = \lceil 2\pi/\alpha \rceil,$$
$$\bar{\mu}(\alpha) := \min\{k \in \mathbb{Z} \mid 2\bar{\beta}(\alpha) \cdot k \geq 2\pi\} = \lceil 2\pi/(\pi - \alpha) \rceil,$$

respectively. Observe that $\hat{\mu}$ is nonstrictly decreasing and $\bar{\mu}$ is nonstrictly increasing. The nonempty sets of the form $\hat{\mu}^{-1}(i)$ for integers i form a partition of $(0, \pi)$. Similarly, the nonempty sets of the form $\bar{\mu}^{-1}(j)$ for integers j form a partition of $(0, \pi)$. The canonical common refinement of these two partitions consists of all nonempty sets of the form $A_{ij} := \hat{\mu}^{-1}(i) \cap \bar{\mu}^{-1}(j)$. Call a pair of integers (i', j') *minimal* if $A_{i'j'} \neq \emptyset$ but $A_{i'-1,j'} = \emptyset$ and $A_{i',j'-1} = \emptyset$. To find all minimal pairs, we plot the solutions of the equations $2\pi/\alpha = i$ and $2\pi/(\pi - \alpha) = j$:

$$\hat{\alpha}_i := 2\pi/i, \qquad \bar{\alpha}_j := \pi - 2\pi/j.$$



The cells of the two partitions of $(0, \pi)$ are as follows:

$$\hat{\mu}^{-1}(i) = [\hat{\alpha}_i, \hat{\alpha}_{i-1}) \text{ for } i = \dots, 5, 4, 3,$$
$$\bar{\mu}^{-1}(j) = (\bar{\alpha}_{j-1}, \bar{\alpha}_j] \text{ for } j = 3, 4, 5, \dots$$

The following lemma is proved by tracing the above plot of points.

Lemma 21. *For any pair (i, j) such that $A_{ij} \neq \emptyset$ there exists a minimal pair (i', j') such that $i' \leq i$ and $j' \leq j$. The full list of minimal pairs (i', j') and their corresponding cells are as follows:*

$$(i', j') = (6, 3), (4, 4), (3, 6);$$
$$A_{6,3} = [\hat{\alpha}_6, \bar{\alpha}_3] = \{\pi/3\},\ A_{4,4} = [\hat{\alpha}_4, \bar{\alpha}_4] = \{\pi/2\},\ A_{3,6} = [\hat{\alpha}_3, \bar{\alpha}_6] = \{2\pi/3\}.$$

Now we can give an explicit description of the set Q. The notation $(i', j') \leq (i, j)$ or $(i, j) \geq (i', j')$ will mean "$i' \leq i$ and $j' \leq j$".

Lemma 22.

$$Q = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid \exists (i', j') \in \mathbb{Z} \times \mathbb{Z} \quad (i', j') \text{ is minimal and } (i', j') \leq (i, j)\}$$
$$= \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid (i, j) \geq (6, 3) \text{ or } (i, j) \geq (4, 4) \text{ or } (i, j) \geq (3, 6)\}.$$

Proof.

$$
\begin{aligned}
(i,j) \in Q \Leftrightarrow \quad & \exists \alpha \in (0,\pi) \quad \alpha i \geq 2\pi \text{ and } 2\bar{\beta}(\alpha)\, j \geq 2\pi \\
\Leftrightarrow \quad & \exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i \text{ and } \bar{\mu}(\alpha) \leq j \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \\
& \text{and } [\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i' \text{ and } \bar{\mu}(\alpha) \leq j'] \\
& \text{and } \neg[\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i'-1 \text{ and } \bar{\mu}(\alpha) \leq j'] \\
& \text{and } \neg[\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i' \text{ and } \bar{\mu}(\alpha) \leq j'-1] \Big) \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \\
& \text{and } [\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i' \text{ and } \bar{\mu}(\alpha) \leq j'] \\
& \text{and } [\forall \alpha \in (0,\pi) \quad i'-1 < \hat{\mu}(\alpha) \text{ or } j' < \bar{\mu}(\alpha)] \\
& \text{and } [\forall \alpha \in (0,\pi) \quad i' < \hat{\mu}(\alpha) \text{ or } j'-1 < \bar{\mu}(\alpha)] \Big) \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \\
& \text{and } [\exists \alpha \in (0,\pi) \quad i'-1 < \hat{\mu}(\alpha) \leq i' \text{ and } j'-1 < \bar{\mu}(\alpha) \leq j'] \\
& \text{and } [\forall \alpha \in (0,\pi) \quad i'-1 < \hat{\mu}(\alpha) \text{ or } j' < \bar{\mu}(\alpha)] \\
& \text{and } [\forall \alpha \in (0,\pi) \quad i' < \hat{\mu}(\alpha) \text{ or } j'-1 < \bar{\mu}(\alpha)] \Big) \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \text{ and } \hat{\mu}^{-1}(i') \cap \bar{\mu}^{-1}(j') \neq \emptyset \\
& \text{and } \neg[\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i'-1 \text{ and } \bar{\mu}(\alpha) \leq j'] \\
& \text{and } \neg[\exists \alpha \in (0,\pi) \quad \hat{\mu}(\alpha) \leq i' \text{ and } \bar{\mu}(\alpha) \leq j'-1] \Big) \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \text{ and } \hat{\mu}^{-1}(i') \cap \bar{\mu}^{-1}(j') \neq \emptyset \\
& \text{and } \Big[\bigcup_{i'' \leq i'-1} \hat{\mu}^{-1}(i'') \Big] \cap \Big[\bigcup_{j'' \leq j'} \bar{\mu}^{-1}(j'') \Big] = \emptyset \\
& \text{and } \Big[\bigcup_{i'' \leq i'} \hat{\mu}^{-1}(i'') \Big] \cap \Big[\bigcup_{j'' \leq j'-1} \bar{\mu}^{-1}(j'') \Big] = \emptyset \Big) \\
& \text{(because } \hat{\mu} \text{ is decreasing and } \bar{\mu} \text{ is increasing)} \\
\Leftrightarrow \quad & \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \text{ and } \hat{\mu}^{-1}(i') \cap \bar{\mu}^{-1}(j') \neq \emptyset \\
& \text{and } \hat{\mu}^{-1}(i'-1) \cap \bar{\mu}^{-1}(j') = \emptyset \text{ and } \hat{\mu}^{-1}(i') \cap \bar{\mu}^{-1}(j'-1) = \emptyset \Big)
\end{aligned}
$$

$$\Leftrightarrow \quad \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \text{ and } A_{i'j'} \neq \emptyset \text{ and } A_{i'-1,j'} = \emptyset \text{ and } A_{i',j'-1} = \emptyset \Big)$$

$$\Leftrightarrow \quad \exists (i',j') \in \mathbb{Z}\times\mathbb{Z} \quad \Big((i',j') \leq (i,j) \text{ and } (i',j') \text{ is minimal} \Big).$$

The second equality in the lemma follows from Lemma 21. □

6.4. **Nonpositive curvature implies nondegeneracy.**

Theorem 23. *Given an orientable product structure or substructure Π, equip Y_Π is with the structure of an M_0-polyhedral complex by making one choice of the angle α uniformly for all triangles in Y_Π. If Y_Π has curvature ≤ 0, then Π is nondegenerate.*

Proof. Suppose that Π is degenerate, so that there is a pair of distinct elements $a, a' \in A$ and the corresponding distinct pair of edges in X_Π labeled a and a' such that the loop in X_Π at x_1 with label $a_i^{-1}a_{i'}$ represents the identity element in the universal group $G_\Pi = \pi_1(X_\Pi)$. By Lemma 5, the edge-loop in $\bar{Y}_\Pi$ with label $a^{-1}a'$ is nullhomotopic in $\bar{X}_\Pi$. Since the product structure is orientable, by Lemma 4 the quotient map $X_\Pi \to Y_\Pi$ induces an isomorphism on the fundamental groups, so we have similar two edges a and a' and a nullhomotopy of the loop $a^{-1}a'$ in the complex Y_Π. Lifting the edges a and a' to edges $\tilde{a}$ and $\tilde{a}'$ in the universal covering $\tilde{Y}_\Pi$ starting at the same point, we obtain the path labeled $\tilde{a}^{-1}\tilde{a}'$. Since the nulhomotopy of the loop $a^{-1}a'$ can also be lifted to $\tilde{Y}_\Pi$, we see that the path $\tilde{a}^{-1}\tilde{a}'$ is also a loop, in $\tilde{Y}_\Pi$.

Since Y_Π has nonpositive curvature, the Cartan-Hadamard theorem (Theorem 7) implies that $\tilde{Y}_\Pi$ is a CAT(0)-space with respect to the induced metric. Lemma 17 says that the inclusion maps of the edges a and a' into Y_Π are local isometric embeddings. Since the interval is simply connected, Theorem 13 says that the edges labeled $\tilde{a}$ and $\tilde{a}'$ are embedded into $\tilde{Y}_\Pi$ isometrically, so they provide two geodesic path with the same endpoints. Since geodesics are unique in $\tilde{Y}_\Pi$ by Theorem 7, we deduce that $\tilde{a} = \tilde{a}'$, hence $a = a'$ in Y_Π. This gives a contradiction with the assumptions. □

7. Necessary and sufficient combinatorial conditions on product structures

We will now see how certain combinatorial conditions on a product structure Π, and its corresponding taiko, relate to geometric structures on the complex Y_Π.

7.1. **Conditions on product structures.**
Consider the following conditions on a product structure Π:

- Orientation. This condition was defined in section 2.4: there exists an orientation $O : \bar{E}_{AB} \to E_{AB}$ on Π. Each of the conditions below will be defined under the assumption that orientation holds.
- No-fold. *A fold* in a taiko is a pair of horizontal edges that are incident to the same vertex $v \in A \sqcup B$, have the same color and the same direction at v, meaning that they are either both incoming towards v or both outgoing from v. (This is related to the notion of folds introduced by Stallings [33].) The no-fold condition says that at each vertex v in the taiko for Π, there is no fold at any vertex in the taiko.

- $\mathsf{No\text{-}pattern}$. *A pattern* in a taiko is an unordered pair of colors of horizontal edges together with their orientations, that occur incident at a common vertex v in $\mathsf{L}_{AB} = \mathsf{L}_A \sqcup \mathsf{L}_B$. That is, a pattern is a pair of the form $\{(c_1, d_1), (c_2, d_2)\}$, where c_1 and c_2 are colors and $d_1, d_2 \in \{in, out\}$. The $\mathsf{no\text{-}pattern}$ condition says that a given product structure has no *repeating* patterns. That is, each pattern occurs at most once in L_{AB}.
- $\mathsf{girth}(p, q)$. *The girth* of a graph $\mathcal{G}$, denoted girth$(\mathcal{G})$, is the length of the shortest nonconstant injective loop in $\mathcal{G}$, where each edge is considered to be of length 1. *The half-girth* of a graph $\mathcal{G}$, denoted half-girth$(\mathcal{G})$, is, naturally, girth$(\mathcal{G})/2$. The condition $\mathsf{girth}(p, q)$ says that girth$(\mathsf{L}_{AB}) \geq p$ and half-girth$(\mathsf{L}_1) \geq q$.
- $\mathsf{girth}(6, 3)(4, 4)(3, 6)$. Also called *the triple girth condition*, it is defined to be the disjunction "$\mathsf{girth}(6, 3)$ or $\mathsf{girth}(4, 4)$ or $\mathsf{girth}(3, 6)$".
- $\mathsf{metric\text{-}girth}(2\pi)$. This condition is a metric version of $\mathsf{girth}(p, q)$. *The metric girth* of a metric graph $\mathcal{G}$, denoted metric-girth$(\mathcal{G})$, is the length of the shortest nonconstant injective loop in G, with respect to the given metric on $\mathcal{G}$. The $\mathsf{metric\text{-}girth}(2\pi)$ condition says that there exists a metric on Y_Π that makes it an M_0-complex such that metric-girth$(\mathsf{L}_{AB}) \geq 2\pi$ and metric-girth$(\mathsf{L}_1) \geq 2\pi$.

These conditions can be verified – algorithmically, in finite time – for any given product structure Π, and for its corresponding taiko. All these conditions, for example, the existence of orientation and the absence of folds, can be checked by a human visually from Fig. 1 and Fig. 2, which is one benefit of drawing taikos in the first place.

Lemma 24. *Let Π be an orientable product structure or substructure. Then, the half-girth of the middle link L_1 is an integer.*

Proof. Let C be the set of colors, that is, equivalence classes of horizontal edges in the taiko for Π. Then C is a bijective copy of the set of middle edges in Y_Π. The set of vertices of the middle link can be naturally identified with the set $A \sqcup (C \times \{in, out\}) \sqcup B$, and the three parts of this union can be viewed as three levels: bottom, middle, and top. The edges in the middle link can only go between the bottom and the middle, or between the middle and the top. This implies that any simple loop (that is, an nonconstant, injective loop) in L_1 traverses an even number of edges. Therefore, half-girth(L_1) is an integer. $\square$

Lemma 25. *Let Π be an orientable product structure or substructure. Then, the following statements are equivalent.*

- *There is a fold in the taiko for Π.*
- *There are double edges in the middle link L_1, i.e., there is a pair of distinct edges L_1 incident to the same pair of vertices in L_1.*
- half-girth$(\mathsf{L}_1) \leq 1$.

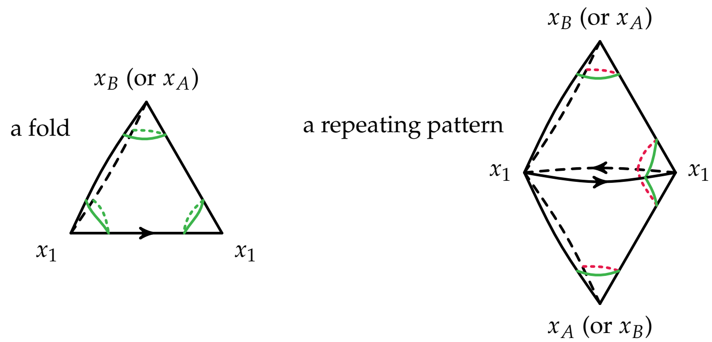
By negation, the following statements are equivalent.

- *The* $\mathsf{no\text{-}fold}$ *condition holds.*
- *There are no double edges in the middle link L_1.*
- half-girth$(\mathsf{L}_1) > 1$ *(or, equivalently, ≥ 2).*

Also, the following statements are equivalent.

- *The conditions* no-fold *and* no-pattern *hold.*
- half-girth$(\mathsf{L}_1) > 2$ *(or, equivalently, ≥ 3).*

Proof. The proof is an exercise using the definition of Y_Π and the following illustrations.



□

Lemma 26 (Necessary conditions for curvature ≤ 0)**.** *Let Π be an orientable product structure or substructure. Put the metric on the 2-complex Y_Π corresponding to an angle α as in section 6.2, and suppose that this metric is of curvature ≤ 0.*

(a) *If $\alpha = \pi/3$, then* no-fold, no-pattern, girth$(6,3)$ *hold.*
(b) *If $\alpha = \pi/4$, then* no-fold, no-pattern, girth$(4,4)$ *hold.*
(c) *If $\alpha = 2\pi/3$, then* no-fold, no-pattern, girth$(3,6)$ *hold.*

Since we are mostly interested in sufficient conditions for nonpositive curvature, we leave the proof of this lemma as an exercise: first show that the curvature ≤ 0 assumption implies the link condition for Y_Π, which in turn implies each of the conditions listed in the lemma.

In Figure 1 and Figure 2, nondegeneracy of the product structure and being torsion-free for the full universal group $\bar{G}_\Pi$ are not immediately apparent, and generally there should be no easy way to verify them. Unless, that is, a given product structure satisfies some favorable sufficient conditions.

Theorem 27 (Sufficient conditions for counterexamples to Kaplansky conjectures)**.** *For the conjunctions*

(1) orientation *and* girth$(6,3)$,
(1′) orientation, no-fold, no-pattern, *and* girth$(\mathsf{L}_{AB}) \geq 6$,
(2) orientation *and* girth$(6,3)(4,4)(3,6)$,
(2′) orientation, no-fold *and* girth$(6,3)(4,4)(3,6)$,
(3) orientation *and* metric-girth(2π),
(3′) orientation, no-fold *and* metric-girth(2π),

the following implications hold: $(1) \Leftrightarrow (1')$, $(2) \Leftrightarrow (2')$, $(3) \Leftrightarrow (3')$, $(1) \Rightarrow (2) \Rightarrow (3)$.

If a product structure Π of size (m,n) satisfies at least one of the conjunctions (1), (1′), (2), (2′), (3), (3′), *then Π is non-degenerate and both universal groups G_Π and $\bar{G}_\Pi$ are torsion-free. In particular, if $m \geq 2$ and $n \geq 2$, then the associated elements a_Π and b_Π in $\mathbb{Z}_2\bar{G}_\Pi$ give a counterexample to the unit conjecture when mn is odd, and a counterexample to the zero-divisor conjecture when mn*

is even. If, in addition, the product structure admits a signature, then the associated elements a_Π and b_Π in $R\bar{G}_\Pi$ give such counterexamples over any ring R with unity.

Proof. The implications (2) $\Leftarrow$ (2′), (3) $\Leftarrow$ (3′) and (1) $\Rightarrow$ (2) are obvious.

The equivalence (1) $\Leftrightarrow$ (1′) follows from Lemma 25: "no-fold and no-pattern" is equivalent to half-girth$(\mathsf{L}_1) \geq 3$.

To prove (2) $\Rightarrow$ (2′), suppose that (2) holds but (2′) does not. Then there is a fold in the taiko, then by Lemma 25, half-girth$(\mathsf{L}_1) \leq 1$, which contradicts the condition girth$(6,3)(4,4)(3,6)$ in (2). This shows the equivalence (2) $\Leftrightarrow$ (2′).

The equivalence (3) $\Leftrightarrow$ (3′) is proved similarly: if there is a fold, then by Lemma 25, there are double edges in L_1 and two such edges form an simple loop of length $\leq 2\beta < 2\pi$, which contradicts condition metric-girth(2π) in (3).

To prove (2) $\Rightarrow$ (3), first assume that girth$(6,3)$ is satisfied. Put the metric structure on Y_Π corresponding to $\alpha = \pi/3$ as in section 6.2. Since any simple loop in L_{AB} has at least 6 edges, then its metric length is at least $6\alpha = 2\pi$. Similarly, since half-girth$(\mathsf{L}_1) \geq 3$, then any simple loop in L_1 has at least 6 edges, so its metric length is at least $6\beta = 3(\pi - \alpha)/2 = 2\pi$. This proves that Y_Π is an M_0-complex of nonpositive curvature. The cases girth$(4,4)$ and girth$(3,6)$ are handled similarly.

To show that any one of the conjuctions (1), (1′), (2), (2′), (3), (3′) implies the existence of units and zero-divisors, it suffices to prove this for the conjunction (3) only. Assume that (3) holds, this implies that the link condition holds for Y_Π, so by Theorem 18, Y_Π is nonpositively curved. By Theorem 10, the universal group $G_\Pi = \pi_1(Y_\Pi)$ is torsion-free, then by Theorem 10 the full universal group $\bar{G}_\Pi = \pi_1(\bar{Y}_\Pi)$ is torsion-free as well. (Another way of proving this is to observe that the metric structure of nonpositive curvature on Y_Π induces a metric structure of nonpositive curvature on its quotient $\bar{Y}_\Pi$, then to apply Theorem 10 to $\bar{Y}_\Pi$.)

By Theorem 23, Π is nondegenerate. If $m \geq 2$ and $n \geq 2$, then by lemmas 1 and 2, the associated units or zero-divisors are nondegenerate. □

Remark 1. As one can see from Theorem 27, the no-fold condition can be removed from (2′) and (3′), but it is helpful for computation: if no-fold is satisfied, then there are no double edges in the middle link L_1, which means that the edges in the middle link can be coded simply as ordered pairs of vertices in L_1. Among the six conjunctions, the conjunctions (1′), (2′) and (3′) are the most suitable ones for utilizing computer search.

Remark 2. Consider all metric structures on Y_Π given by all choices of angles α, as in section 6.2. For each pair (i,j) in the set Q defined in section 6.3 there exists $\alpha \in (0,\pi)$ such that girth(i,j) implies metric-girth(2π), that is, the link condition. Lemma 22 then implies that the triple girth condition girth$(6,3)(4,4)(3,6)$ is the weakest possible condition on girth one can hope for to look for product structures whose complex Y_Π satisfies the link condition (for some choice of α). That is, the triple girth condition maximizes the chances of finding complexes Y_Π of this type, and their associated units and zero-divisors.

7.2. **The program.** To look for counterexamples to the Kaplansky unit and zero-divisor conjectures, follow these steps.

- For each size (m, n), search for product structures of size (m, n) satisfying at least one conjunction in Theorem 27.
- If such product structures are found, list and classify such product structures and their associated units or zero-divisors (a_Π, b_Π).
- If not found for a given size (m, n), conclude that for the metric structures on complexes Y_Π associated with any product structures of size (m, n) and any choices of α are not of curvature ≤ 0.
- Modify the construction of the complex Y_Π and the conditions in a way that they still imply that the full universal group $\bar{G}_\Pi$ is torsion-free and the product structure Π is nondegenerate.
- Repeat.

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