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NON-MICROSTATES FREE ENTROPY DIMENSION FOR GROUPS

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Abstract. We show that for any discrete finitely-generated group G and any self-adjoint n-tuple $X_1, \dots, X_n$ of generators of the group algebra $\mathbb{C}G$, Voiculescu's non-microstates free entropy dimension $\delta^*(X_1, \dots, X_n)$ is exactly equal to $\beta_1(G) - \beta_0(G) + 1$, where β_i are the ℓ^2-Betti numbers of G.

1 Introduction

In [V1], using ideas from his theory of free entropy and free probability, D. Voiculescu has associated to every n-tuple of self-adjoint elements $(X_1, \dots, X_n)$ in a tracial von Neumann algebra a number $\delta(X_1, \dots, X_n)$, which he called the free entropy dimension of this n-tuple. The free entropy dimension is, very roughly, a kind of asymptotic Minkowski dimension of the set of n-tuples of matrices that approximate the variables $X_1, \dots, X_n$ in non-commutative moments (these are commonly known as "sets of microstates", see [V2,4], [J] for further details).

It is hoped that this number is an invariant of the von Neumann algebra generated by $X_1, \dots, X_n$. While this hope is presently out of reach in the most interesting cases, this quantity has played a key role in the solution of several long-standing von Neumann algebra problems (see, e.g. [V4] for a survey).

Nonetheless, it is known that a certain technical modification of δ, δ_0 depends only on the algebra generated by $X_1, \dots, X_n$ (and the ambient trace). In particular, if we start with a discrete finitely-generated group G, then δ_0, evaluated on any set of generators of G, gives the same number,

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which is an invariant of G. This invariant is quite mysterious, and its exact value is known in only a few cases (such as free products of abelian groups).

In [V3], Voiculescu has further introduced a different approach to free entropy and free entropy dimension, based on the theory of free Hilbert transform. This "microstates-free" approach has resulted in two definitions of "non-microstates" free entropy dimension-like quantities, δ^* and $\delta^\star$. While it is suspected that $\delta^* = \delta^\star$, we only know that always $\delta^\star \geq \delta^*$. By a deep result of Biane, Capitaine and Guionnet [BiCG], $\delta^* \geq \delta$.

Much less is known about δ^* than about δ; in all the known cases they assume the same value, although this statement speaks more for the small number of cases in which the value of both is known than for the existence of a general strategy to prove that they are the same for some class of n-tuples. Only recently have there been any non-trivial computations of δ^* ([A],[S]).

Let G be a finitely generated discrete group, and let $\mathbb{C}G$ be its group algebra, endowed with the involution $\left(\sum_\gamma \alpha_\gamma \gamma\right)^* = \sum_\gamma \bar{\alpha}_\gamma \gamma^{-1}$ and the tracial linear functional $\tau\left(\sum_\gamma \alpha_\gamma \gamma\right) = \alpha_e$. Let $X_1, \dots, X_n$ be any generators of this algebra, which are self-adjoint (e.g. if $\gamma_1, \dots, \gamma_m$ are generators of G one could take $n = 2m$ and $X_j = \gamma_j + \gamma_j^{-1}$, $1 \leq j \leq m$, $X_j = -i(\gamma_{j-m} - \gamma_{j-m}^{-1})$, $j = m+1, \dots, 2m$.

Recently, in [CoS] A. Connes and the second author have proved that

$$\delta^*(X_1, \dots, X_n) \leq \delta^\star(X_1, \dots, X_n) \leq \beta_1(G) - \beta_0(G) + 1\,, \tag{1.1}$$

where $\beta_i(G)$ are Atiyah's ℓ^2-Betti numbers of the group G (see [At], [CG], [L2]). The appearance of ℓ^2-invariants of G in connection with free entropy dimension has been conjectured by specialists ever since the fundamental work of Gaboriau [G1,2]. Nonetheless, this connection remains quite surprising to us, since free entropy dimension relies on the notion of free Brownian motion, while ℓ^2-Betti numbers are homological in nature, and it is hard to say why the two must have anything in common.

The main result of this paper is that in fact equality holds: we prove that

$$\delta^*(X_1, \dots, X_n) = \delta^\star(X_1, \dots, X_n) = \beta_1(G) - \beta_0(G) + 1\,,$$

for any finitely-generated group G and any set of self-adjoints $X_1,\dots,X_n \in \mathbb{C}G$ generating $\mathbb{C}G$. In particular, we conclude that in this case, $\delta^* = \delta^\star$, and both are algebraic invariants.

The main technical tool is a result showing that arbitrary ℓ^2 1-coboundaries on the Cayley graph of G can be approximated in ℓ^2 norm by coboundaries of the form δg, where $g \in \ell^\infty(G)$. This result holds more generally for arbitrary graphs, and for ℓ^2 replaced by ℓ^p, $1 \leq p < \infty$.

Using this result, we utilize a lower estimate for non-microstates free entropy dimension from [S], which combined with (1.1) gives the main result.

Notation. Throughout this paper, G will denote a finitely generated discrete group. We write $\ell^2(G)$ for the Hilbert space of square-summable functions on G. We denote by λ and ρ the left- and right-regular representation of G on $\ell^2(G)$, and by $L(G)$ the group von Neumann algebra, which is the weak operator topology closure of the linear span of the image $\lambda(G)$ viewed as a subalgebra of the algebra of bounded operators $B(\ell^2(G))$. By τ we shall always denote the von Neumann trace on $L(G)$ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$, where δ_e is the delta function at the identity of G. The restriction of τ to $\lambda(\mathbb{C}G)$ is the canonical group trace on the group algebra determined by linearity and the condition $\tau(g) = 1$ if $g = e$, and $\tau(g) = 0$ if $g \neq e$.

The letter M will denote a general von Neumann algebra with a normal (i.e. weak-operator continuous) tracial state $\tau : M \to \mathbb{C}$, $\tau(xy) = \tau(yx)$. The von Neumann algebra M acts by left and right multiplication on the Hilbert space $L^2(M)$, which is the completion of M in the norm $\|m\|_2 = \tau(m^*m)^{1/2}$. In the case that $M = L(G)$, $L^2(M) = \ell^2(G)$, and the left and right actions of $L(G)$ on this space extend the left and right actions of G. We will denote by M^o the opposite von Neumann algebra. The letter J will denote the anti-linear Tomita conjugation operator $J : L^2(M) \to L^2(M)$ extending $J(m) = m^*$. The operator J satisfies the property that for $x \in M$ and $\xi \in L^2(M)$, $JxJ\ \xi = \xi\ x^*$, i.e. it switches the right and left actions of M. In particular, for any $x \in M$, JxJ commutes with M.

If $H \subset L^2(M)^{\oplus n}$ is a closed M-submodule of a multiple of the left module $L^2(M)$, we denote by $\dim_M H$ its Murray–von Neumann dimension. This dimension satisfies the usual monotonicity and additivity properties (see Chapter X in [MN] (especially Theorem X on p. 182), or, for a more accessible introduction, [GoHJ], [CG]).

We denote by $B(L^2(M))$ the space of all bounded linear operators on $L^2(M)$. Finally, we will denote by HS the space of Hilbert–Schmidt operators $T : L^2(M) \to L^2(M)$, i.e. the operators $T \in B(L^2(M))$ for which the norm $\|T\|_{HS} = \mathrm{Tr}(T^*T)$ is finite. HS is a Hilbert space with the inner product $\langle T, S\rangle = \mathrm{Tr}(TS^*)$. HS can be identified with the Hilbert space tensor product $L^2(M)\bar{\otimes}L^2(M^o)$ by the map $m \otimes n^o \mapsto mP_1n$, where $P_1 \in HS$ denotes the rank one projection onto the vector $1 \in M \subset L^2(M)$, $m \in M$ and $n^o \in M^o$. By definition, the von Neumann algebra tensor product

$M\bar{\otimes}M^o$ acts on the tensor product Hilbert space $L^2(M)\bar{\otimes}L^2(M^o)$ and thus on HS. The action of the algebraic tensor product $M\otimes M^o \subset M\bar{\otimes}M^o$ on HS is explicitly given by $(m\otimes n^o)\cdot T = mTn$ (composition of operators on $L^2(M)$), for $T\in HS$, $m\in M$ and $n^o\in M^o$.

2 Approximation of ℓ^p-Summable 1-Coboundaries on Graphs

Let $\mathcal{G}$ be a graph. $C^i(\mathcal{G},\mathbb{R})$ will denote the set of real i-cochains on $\mathcal{G}$, without any assumptions on their support. For each $1\le p\le\infty$, let $C^i_{(p)}(\mathcal{G},\mathbb{R})$ be the set of elements in $C^i(\mathcal{G},\mathbb{R})$ which have finite ℓ^p norm. Finally, let $\delta: C^0(\mathcal{G},\mathbb{R})\to C^1(\mathcal{G},\mathbb{R})$ be the coboundary map.

Theorem 2.1. *Let $\mathcal{G}$ be an arbitrary graph, $p\in[1,\infty)$ and $f\in C^0(\mathcal{G},\mathbb{R})$ be such that $\delta f\in C^1_{(p)}(\mathcal{G},\mathbb{R})$. Then for each $\varepsilon>0$ there exists $g\in C^0_{(\infty)}(\mathcal{G},\mathbb{R})$ such that $\|\delta f-\delta g\|_p<\varepsilon$. In particular, $\delta g\in C^1_{(p)}(\mathcal{G},\mathbb{R})$.*

The same result holds in the complex-valued case.

Proof. Let Σ_i denote the set of i-simplices in $\mathcal{G}$. We are given a function $f:\Sigma_0\to\mathbb{R}$ such that $\delta f:\Sigma_1\to\mathbb{R}$ is ℓ^p-summable.

Fix some $t\in[0,\infty)$, denote $U_t=f^{-1}([-t,t])\subseteq\Sigma_0$ and for $x\in\Sigma_0$,

$$f_t(x)=\begin{cases}-t & \text{if } f(x)\in(-\infty,-t)\\ f(x) & \text{if } f(x)\in[-t,t]\\ t & \text{if } f(x)\in(t,\infty)\,.\end{cases}$$

Obviously, $|f_t(x)|\le\min\{|f(x)|,t\}\le t$, so in particular $f_t\in C^0_{(\infty)}(\mathcal{G},\mathbb{R})$ for each t.

Let δU_t be the set of all edges in $\mathcal{G}$ all of whose incident vertices are in U_t. We have

$$\bigcup_{t\in[0,\infty)} U_t=\Sigma_0\,,$$

and therefore

$$\bigcup_{t\in[0,\infty)} \delta U_t=\Sigma_1\,, \tag{2.1}$$

where $\{\delta U_t\}$ is an increasing sequence of sets.

Since f and f_t coincide on U_t, then δf and δf_t coincide on δU_t, that is

$$\operatorname{supp}(\delta f-\delta f_t)\subseteq\Sigma_1\setminus\delta U_t\,. \tag{2.2}$$

We need the following lemma.

Lemma 2.2. *With the above notation,*

$$\big|\delta f_t(e)\big|\le\big|\delta f(e)\big|\quad \textit{for all } t\in[0,\infty) \textit{ and } e\in\Sigma_1\,.$$

Proof. Since δf and δf_t coincide on δU_t, it only remains to show the inequality when $e \in \Sigma_1 \backslash \delta U_t$, that is when the edge e is incident to a vertex x in $\Sigma_0 \backslash U_t$. By the definition of U_t this means that $f(x) \in (-\infty, -t) \cup (t, \infty)$. We can assume $f(x) \in (t, \infty)$, the opposite case can be done similarly. Let x' be the other incident vertex of e. There are three obvious cases to consider for x', and we use the definition of f_t in each case.

If $f(x') \in (t, \infty)$ then

$$\left|\delta f_t(e)\right| = \left|f_t(x) - f_t(x')\right| = |t - t| = 0 \leq \left|\delta f(e)\right| .$$

If $f(x') \in [-t, t]$ then

$$\begin{aligned}\left|\delta f_t(e)\right| &= \left|f_t(x) - f_t(x')\right| = \left|t - f(x')\right| = t - f(x') \\ &\leq f(x) - f(x') = \left|f(x) - f(x')\right| = \left|\delta f(e)\right| .\end{aligned}$$

If $f(x') \in (-\infty, -t)$ then

$$\begin{aligned}\left|\delta f_t(e)\right| &= \left|f_t(x) - f_t(x')\right| = \left|t - (-t)\right| = t + t \\ &\leq f(x) - f(x') = \left|f(x) - f(x')\right| = \left|\delta f(e)\right| .\end{aligned}$$

This finishes the proof of the lemma. □

Now we can finish the proof of Theorem 2.1. Since δf is ℓ^p-summable, given any $\varepsilon > 0$, (2.1) guarantees the existence of $t \in [0, \infty)$ such that

$$\|\delta f|_{\Sigma_1 \backslash \delta U_t}\|_p < \varepsilon/2 ,$$

then by Lemma 2.2,

$$\|\delta f_t|_{\Sigma_1 \backslash \delta U_t}\|_p \leq \|\delta f|_{\Sigma_1 \backslash \delta U_t}\|_p < \varepsilon/2 ,$$

so (2.2) implies that

$$\begin{aligned}\|\delta f - \delta f_t\|_p &= \|(\delta f - \delta f_t)|_{\Sigma_1 \backslash \delta U_t}\|_p \\ &= \|\delta f|_{\Sigma_1 \backslash \delta U_t} - \delta f_t|_{\Sigma_1 \backslash \delta U_t}\|_p \leq \varepsilon/2 + \varepsilon/2 = \varepsilon .\end{aligned}$$

Setting $g = f_t$ completes the proof of Theorem 2.1 in the real case. The complex case is obtained by separately approximating the real and imaginary parts of δf. □

3 ℓ^2-Betti Numbers

3.1 ℓ^2-Betti numbers for groups. The notion of ℓ^2-Betti numbers for groups goes back to Atiyah [At] and Cheeger and Gromov [CG]. We refer the reader to the book [L2] for more details and only sketch the construction here.

Assume that the group G acts freely on a CW-complex X, and that the complex X is "co-finite" (i.e. for each dimension i there is a finite number of i-cells in X, so that every other i-cell in X can be obtained from one of them by the group action). Let $C_i^{(2)}(X,\mathbb{C})$ denote the complex Hilbert space whose orthonormal basis is formed by the i-cells of the complex X. Then G acts on $C_i^{(2)}(X)$; this action of course extends to a representation of the group algebra $\mathbb{C}G$ of G on this Hilbert space. This representation is contained in a multiple of the left-regular representation, and hence the action of $\mathbb{C}G$ extends by continuity to an action of the group von Neumann algebra $L(G)$.

Thus one can speak of the Murray–von Neumann dimension of any closed G-invariant subspace of the Hilbert space $C_i^{(2)}(X)$.

The boundary maps ∂_i of the complex X extend to continuous linear operators $\hat{\partial}_i : C_i^{(2)}(X,\mathbb{C}) \to C_{i-1}^{(2)}(X,\mathbb{C})$.

The reduced ℓ^2-homology of the complex X is defined to be the sequence of Hilbert spaces

$$H_k^{(2)}(X) = \ker \hat{\partial}_k / \overline{\operatorname{im} \partial_{k+1}}\,,$$

where closure is taken with respect to the Hilbert space norm (the closure of $\operatorname{im}\partial_{k+1}$ is the same as that of $\operatorname{im}\hat{\partial}_{k+1}$). Note that $H_k^{(2)}(X)$ can be thought of as the orthogonal complement of $\operatorname{im}\partial_{k+1}$ inside $\ker\hat{\partial}_k \subset C_k^{(2)}(X,\mathbb{C})$. Thus one can consider its Murray–von Neumann dimension, which is exactly the k-th ℓ^2-Betti number of (X,G),

$$\beta_k(X,G) = \dim_{L(G)} H_k^{(2)}(X)\,.$$

In the case that X is not co-finite, one writes X as an increasing union of co-finite G-invariant subcomplexes X_n, $n = 1,2,\dots$. In that case the ℓ^2-Betti numbers can be computed as the following limits:

$$\beta_k(X,G) = \sup_n \inf_{m\geq n} \dim_{L(G)} \frac{\ker \hat{\partial}_k : C_k^{(2)}(X_n,\mathbb{C}) \to C_{k-1}^{(2)}(X_n,\mathbb{C})}{\overline{(\operatorname{im}\hat{\partial}_{k+1} : C_{k+1}^{(2)}(X_m,\mathbb{C}) \to C_k^{(2)}(X_m,\mathbb{C}))} \cap C_k^{(2)}(X_n,\mathbb{C})}\,. \tag{3.1}$$

(closure in Hilbert space norm, see [CG]).

The main point of interest for us is the fact that if the CW-complex X is n-connected, then the first $n+1$ ℓ^2-Betti numbers $\beta_0(X,G),\beta_1(X,G),\dots,$ $\beta_n(X,G)$ are independent of X and are invariants of the group G. In this case, they are referred to as the ℓ^2-Betti numbers of the group G.

3.2 Zeroth and first ℓ^2-Betti numbers for finitely-generated groups. If (as we are in the present paper) one is only interested in the zeroth and first ℓ^2-Betti numbers of a finitely generated group G, then one can make an explicit choice of a one-connected CW-complex that can be used to compute the first two ℓ^2-Betti numbers.

Let $\mathcal{G}$ denote the Cayley graph of G with respect to the set of generators $g_1, \dots, g_n$. Then G acts on $\mathcal{G}$ by left translation. We view $\mathcal{G}$ as a CW-complex, whose 1-cells are the edges of $\mathcal{G}$ and whose 0-cells are the vertices of $\mathcal{G}$. There exists a simply-connected CW-complex X, whose 1-skeleton is $\mathcal{G}$; it is obtained from $\mathcal{G}$ by gluing in a single 2-cell for each non-trivial loop in $\mathcal{G}$.

The action of G on the CW-complex X need not be co-finite (although it is co-finite when restricted to the 1-skeleton, since the group G is finitely generated). However, one can write X as a union of X_m, $m = 1, 2, \dots$, where X_m are G-invariant subcomplexes of X, having $\mathcal{G}$ as their 1-skeletons, and with the property that each X_m is co-finite. Indeed, one could just enumerate all of the 2-cells used in the construction of X, and for each m, let X_m be the space arising after the first m 2-cells, together with all of their G-translates, are glued to $\mathcal{G}$.

We consider the spaces of i-cells of X_m as subsets $C_i(X_m) \subset C_i(X)$. Let us denote by $C_i^{(2)}(X_m, \mathbb{C})$ the completion of the space $C_i(X_m, \mathbb{C})$ with respect to ℓ^2-norm. Let $\partial_i : C_i(X_m, \mathbb{C}) \to C_{i-1}(X_m, \mathbb{C})$ be the boundary map and $\hat{\partial}_1 : C_1^{(2)}(X_m, \mathbb{C}) \to C_0^{(2)}(X_m, \mathbb{C})$, $i = 1, 2$, be its continuous extension.

In this case [CG], [BV] the first two ℓ^2-Betti numbers of G are defined as the following Murray–von Neumann dimensions over the group von Neumann algebra $L(G)$ of G:

$$\beta_1(G) = \dim_{L(G)} H_1^{(2)}(X) \, , \quad \beta_0(G) = \dim_{L(G)} H_0^{(2)}(X) \, .$$

Then we have by additivity of dimension and by (3.1),

$$\begin{aligned} \beta_1(G) &= \inf_{m \geq 1} \dim_{L(G)} \frac{\ker \hat{\partial}_1 : C_1^{(2)}(X_1, \mathbb{C}) \to C_0^{(2)}(X_1, \mathbb{C})}{\overline{\operatorname{im} \hat{\partial}_2 : C_2^{(2)}(X_m, \mathbb{C}) \to C_1^{(2)}(X_m, \mathbb{C})}} \\ &= \inf_m \big(\dim_{L(G)} \ker \hat{\partial}_1 - \dim_{L(G)} \overline{\hat{\partial}_2(C_2(X_m))} \big) \, , \\ \beta_0(G) &= 1 - \dim_{L(G)} \overline{\operatorname{im} \hat{\partial}_1} \, . \end{aligned} \tag{3.2}$$

Note that $\overline{\partial_2(C_2(X_m))}$, $m = 1, 2, \dots$, are increasing $L(G)$-submodules of a finite-dimensional $L(G)$-module $\ker \hat{\partial}_1$. Thus

$$\inf_m \big(\dim_{L(G)} \ker \hat{\partial}_1 - \dim_{L(G)} \overline{\hat{\partial}_2(C_2(X_m))} \big) = \dim_{L(G)} \ker \hat{\partial}_1 - \dim_{L(G)} \overline{\hat{\partial}_2(C_2(X))} \, .$$

Since X is simply-connected, $\operatorname{im} \partial_2 = \ker \partial_1$ and their ℓ^2-closures inside $C_1^{(2)}$ coincide with the closure of the space $\operatorname{im} \hat{\partial}_2$. Thus

$$\beta_1(G) = \dim_{L(G)} \ker \hat{\partial}_1 - \dim_{L(G)} \overline{\ker \partial_1} \, . \tag{3.3}$$

Denote by $C^i(X,\mathbb{C})$ the space of all cochains on X, i.e. the algebraic dual of $C_i(X,\mathbb{C})$, and by

$$\delta : C^i(X,\mathbb{C}) \to C^{i+1}(X,\mathbb{C})$$

the coboundary map. Let $C^i_{(2)}(X,\mathbb{C})$ be the space of all ℓ^2-summable i-cochains on X. Then by duality,

$$\overline{\ker \partial_1} = \overline{\operatorname{im} \partial_2} = \left\{c \in C^1_{(2)}(X,\mathbb{C}) : \delta c = 0\right\}^{\perp} \subset C_1^{(2)}(X,\mathbb{C}) . \tag{3.4}$$

Here we identify both $C^1_{(2)}(X,\mathbb{C})$ and $C_1^{(2)}(X,\mathbb{C})$ with $\ell^2(\Sigma_1)$, Σ_1 being the set of 1-simplices in X, and all the closures and orthogonal complements are taken in $\ell^2(\Sigma_1)$.

The first cohomology of the complex $C^*(X,\mathbb{C})$ vanishes, since X is simply-connected.

Therefore if $c \in C^1_{(2)}(X,\mathbb{C})$ satisfies $\delta c = 0$, then $c = \delta f$ for some $f \in C^0(X,\mathbb{C})$. Thus by (3.4),

$$\overline{\ker \partial_1} = \left(\delta(C^0(X,\mathbb{C})) \cap C^1_{(2)}(X,\mathbb{C})\right)^{\perp} \subset C_1^{(2)}(X,\mathbb{C}) .$$

Theorem 2.1 says that

$$\delta\big(C^0(X,\mathbb{C})\big) \cap C^1_{(2)}(X,\mathbb{C}) \subseteq \overline{\delta\big(C^0_{(\infty)}(X,\mathbb{C})\big) \cap C^1_{(2)}(X,\mathbb{C})} ,$$

so we get the following corollary:

Corollary 3.1. *The closure of* $\operatorname{im} \partial_2$ *equals*

$$\overline{\ker \partial_1} = \left(\delta(C^0_{(\infty)}(X,\mathbb{C})) \cap C^1_{(2)}(X,\mathbb{C})\right)^{\perp} \subset C_1^{(2)}(X,\mathbb{C}) .$$

Lemma 3.2. *Let* $\delta^{(2)}(G) = \beta_1(G) - \beta_0(G) + 1$. *Then*

$$\delta^{(2)}(G) = n - \dim_{L(G)} \overline{\ker \partial_1} = \dim_{L(G)} \overline{\left(\delta(C^0_{(\infty)}(X,\mathbb{C})) \cap C^1_{(2)}(X,\mathbb{C})\right)} .$$

Proof. We have by (3.3) and (3.2)

$$\begin{aligned}
\beta_1(G) - \beta_0(G) + 1 &= \dim_{L(G)} \ker \hat\partial_1 - \dim_{L(G)} \overline{\ker \partial_1} - 1 + \dim_{L(G)} \overline{\operatorname{im} \hat\partial_1} + 1 \\
&= \dim_{L(G)} \ker \hat\partial_1 + \dim_{L(G)} \overline{\operatorname{im} \hat\partial_1} - \dim_{L(G)} \overline{\ker \partial_1} \\
&= \dim_{L(G)} C_1^{(2)}(X;\mathbb{C}) - \dim_{L(G)} \overline{\ker \partial_1} ,
\end{aligned}$$

the last equality by additivity of Murray–von Neumann dimension. But $C_1^{(2)}(X;\mathbb{C}) \cong (\ell^2(G))^{\oplus n}$, so that

$$\delta^{(2)}(G) = n - \dim_{L(G)} \overline{\ker \partial_1} = \dim_{L(G)} (\ker \partial_1)^{\perp} .$$

It remains to apply Corollary 3.1. $\square$

3.3 Δ and L^2-homology of algebras. Let (M,τ) be a tracial von Neumann algebra, and let $X_1,\dots,X_n \in M$ be a self-adjoint set of elements (i.e. we assume that for each i, there is a j so that $X_i^* = X_j$). Let HS be the space of Hilbert–Schmidt operators on the Hilbert space $L^2(M,\tau)$.

Let $J : L^2(M,\tau) \to L^2(M,\tau)$ be the anti-linear Tomita conjugation operator (see notation). Then JMJ is exactly the commutant of M in $B(L^2(M))$.

We view HS as a bimodule over M using the action

$$(m_1 \otimes m_2^o)\cdot T = m_1 T m_2\,,\quad m_1,m_2 \in M\,,\ T \in HS\,.$$

Note that since $HS \cong L^2(M,\tau)\bar{\otimes}L^2(M,\tau)^o \cong L^2(M\bar{\otimes}M^o)$, the action of $M \otimes M^o$ on HS extends by continuity to the action of the von Neumann algebra $M\bar{\otimes}M^o$, which is exactly the left-multiplication action of $M\bar{\otimes}M^o$ on $L^2(M\bar{\otimes}M^o)$. In particular, if H is any M,M-sub-bimodule of HS, which is closed in the Hilbert–Schmidt norm, then it is a module over $M\bar{\otimes}M^o$; in particular, the Murray–von Neumann dimension of H over $M\bar{\otimes}M^o$ makes sense.

Some of the main ideas of the approach to L^2 homology of algebras in [CoS], when particularized to the case of the first Betti number, can be summarized in the following (well-known) table, giving a dictionary between group and von Neumann algebra terms (here $[X,Y] = XY - YX$ denotes the commutator of X and Y):

Group G	von Neumann algebra M
$\ell^2(G)$ as a group module	HS as an M,M-bimodule
$g_1,\dots,g_n$ generators of G	$X_j = \lambda_{g_j}$, $j = 1,\dots,n$ in the left-regular representation λ of G
$\ell^\infty(G)$	$B(L^2(M,\tau))$
Function f on G	Operator m_f of multiplication by f
$\delta f = (\rho_{g_1}(f) - f,\dots,\rho_{g_n}(f) - f) \in C^1_{(2)}(\mathcal{G}) \cong \ell^2(G)^{\oplus n}$ with $f \in \ell^\infty(G)$ (ρ is the right-regular representation)	$([D,JX_1J],\dots,[D,JX_nJ]) \in HS^n$ for $D \in B(L^2(M,\tau))$ (see equation (3.7) and also Lemma 3.4).

Here $[\,\cdot\,,\cdot\,]$ denotes the commutator in $B(L^2(M))$.

Following the ideas presented in the table above and [S, Corollary 2.12] (we caution the reader that the roles of M and JMJ are switched in the present paper compared to [S]), consider the set

$$H_0(X_1,\dots,X_n)=\big\{(\Xi_1,\dots,\Xi_n)\in HS^n:\exists D\in B(L^2(M))\\ \text{s.t. } \Xi_j=[D,JX_jJ]\ \forall j]\big\}\,.$$

Then H_0 is an M,M-bimodule.

DEFINITION 3.1. Let

$$\underline{\Delta}(X_1,\dots,X_n)=\dim_{M\bar{\otimes}M^o}\overline{H_0(X_1,\dots,X_n)}\,,$$

where the closure is taken in the Hilbert–Schmidt topology on HS.

The quantity $\underline{\Delta}$ has appeared in [S] in connection with some technical estimates on free entropy dimension. As we shall see later in Lemma 3.5 (and as is apparent from our table of analogies), the space $H_0(X_1,\dots,X_n)$ is the von Neumann algebra analog of the space

$$\big\{c\in C^1_{(2)}(\mathcal{G}):c=\delta f\text{ for some }f\in\ell^\infty(G)\big\}\,.$$

The proof of the following lemma was inspired by the work of Bekka and Valette [BV].

LEMMA 3.3. *Assume that $X_1,\dots,X_n$ generate M as a von Neumann algebra. Then $\underline{\Delta}(X_1,\dots,X_n)$ depends only on the algebra $\mathbb{C}(X_1,\dots,X_n)$ generated by $X_1,\dots,X_n$ and the trace τ.*

Proof. For $D\in B(L^2(M))$, define a Hilbert space seminorm by

$$\|D\|_{X_1,\dots,X_n}=\left(\sum_{j=1}^n\|[D,JX_jJ]\|_{HS}^2\right)^{1/2}. \tag{3.5}$$

Let $\tilde{D}(X_1,\dots,X_n)=\{D:\|D\|_{X_1,\dots,X_n}<\infty\}$, and let $D_0(X_1,\dots,X_n)$ be the Hilbert space obtained from $\tilde{D}(X_1,\dots,X_n)$ after separation and completion. Endow $D_0(X_1,\dots,X_n)$ with the M,M-bimodule structure coming from the action $(m\otimes n^o)\cdot D=mDn$. Then the map

$$D\mapsto\big([D,JX_1J],\dots,[D,JX_nJ]\big)$$

descends and extends to an $M\bar{\otimes}M^o$-module isomorphism of $D_0(X_1,\dots,X_n)$ with the Hilbert–Schmidt completion of $H_0(X_1,\dots,X_n)$.

Let $Y_1,\dots,Y_m\in\mathbb{C}(X_1,\dots,X_n)$. By the definition of the seminorm in (3.5) we clearly have

$$\|D\|_{X_1,\dots,X_n}\le\|D\|_{X_1,\dots,X_n,Y_1,\dots,Y_m}\,.$$

Also, since each Y_j is a polynomial in $X_1,\dots,X_n$, $\|[D,JY_jJ]\|_{HS}\le C_j\|D\|_{X_1,\dots,X_n}$ for some constants $C_1,\dots,C_m$. It follows that the norms $\|\cdot\|_{X_1,\dots,X_n}$ and $\|\cdot\|_{X_1,\dots,X_n,Y_1,\dots,Y_m}$ are equivalent. Thus the Hilbert space completions of $H_0(X_1,\dots,X_n)$ and $H_0(X_1,\dots,X_n,Y_1,\dots,Y_m)$ are isomorphic as $M\bar{\otimes}M^o$-modules. Thus

$$\underline{\Delta}(X_1,\dots,X_n)=\underline{\Delta}(X_1,\dots,X_n,Y_1,\dots,Y_m)\,. \tag{3.6}$$

If $Y_1,\ldots,Y_m$ generate $\mathbb{C}(X_1,\ldots,X_n)$, then by (3.6)

$$\underline{\Delta}(Y_1,\ldots,Y_n)=\underline{\Delta}(X_1,\ldots,X_n,Y_1,\ldots,Y_m)=\underline{\Delta}(X_1,\ldots,X_n)\,,$$

as claimed. □

Now let G be a discrete group, $S=\{g_1,\ldots,g_n\}$ a finite symmetric set of generators (so that if $g\in S$, then $g^{-1}\in S$.) Let $\lambda,\rho:G\to B(\ell^2(G))$ be the left- and right-regular representations given by $\lambda_g(f)(h)=f(g^{-1}h)$ and $\rho_g(f)(h)=f(hg)$. Then $J\rho_gJ=\lambda_{g^{-1}}$. Let M be the von Neumann algebra of G.

The following lemma is standard:

Lemma 3.4. *Consider the map $\phi:\ell^\infty(G)^{\oplus n}\to B(\ell^2(G))^n$ given by*

$$\phi(\xi_1,\ldots,\xi_n)=(m_{\xi_1},\ldots,m_{\xi_n})\,,$$

where m_f denotes the operator of pointwise multiplication by $f\in\ell^\infty(G)$. Then

(a) $\phi(\ell^2(G)^{\oplus n})\subset HS^n$*;*

(b) *For any closed G-invariant subspace $V\subset\ell^2(G)^{\oplus n}$, one has*

$$\dim_{M\bar{\otimes}M^o}\overline{M\phi(V)M}=\dim_M V\,.$$

Proof. Part (a) is clear.

For part (b), notice that we can identify $M=L(G)$ with M^o, and also HS with $\ell^2(G)\bar{\otimes}\ell^2(G)=\ell^2(G\times G)$.

With these identifications, if $\xi=\sum_g a_g\delta_g\in\ell^2(G)$, with δ_g denoting the delta function at g, then $\phi(\xi)=\sum a_g\delta_{g\times g}\in\ell^2(G\times G)$. Hence ϕ is exactly the continuous extension to L^2 of the induction map

$$1\otimes\cdot:L(G)\to\big(L(G\times G)\big)\otimes_{L(G)}L(G)=L(G)\bar{\otimes}L(G)\,,$$

corresponding to the diagonal inclusion of G into $G\times G$ (see [L1, Theorem 3.3]). Now (b) follows because induction preserves dimension [L1, Theorem 3.3]. □

Recall that $\delta^{(2)}(G)$ was defined by $\delta^{(2)}(G)=\beta_1(G)-\beta_0(G)+1$.

Lemma 3.5. *Let $S=\{g_1,\ldots,g_n\}$ be a symmetric generating set for G. Let $U_j=\lambda_{g_j}$. Then $\underline{\Delta}(U_1,\ldots,U_n)\geq\delta^{(2)}(G)$, as defined in Lemma 3.2.*

Proof. Let $\mathcal{G}$ be the Cayley graph of G with respect to S. Identify $C^1_{(2)}(\mathcal{G})$ with $\ell^2(G)^{\oplus n}$, by identifying the j-th copy of $\ell^2(G)$ with edges labeled g_j.

For $f\in\ell^\infty(G)$, denote by δ_jf the j-th component of δf in this decomposition. Thus $\delta_jf=\rho_{g_j}(f)-f$. Let

$$a(f)=(\delta_1f,\ldots,\delta_nf)\,.$$

To prove the inequality $\underline{\Delta}\geq\delta^{(2)}$, we need to provide a lower estimate on the dimension of the bimodule $\overline{H_0(U_1,\ldots,U_n)}$, and so we need some way of constructing elements in $H_0(U_1,\dots,U_n)$. In order to do that, we need some way of constructing bounded operators D so that $[D, JU_jJ] \in HS$ for all $j = 1,\dots,n$.

We note that by Lemma 3.2, we have that

$$\delta^{(2)}(G) = \dim_{L(G)} \overline{\left\{c \in C^1_{(2)}(\mathcal{G}) : c = \delta f \text{ for some } f \in \ell^\infty(G)\right\}}\,.$$

Now let $f \in \ell^\infty(G)$ be such that $\delta f \in C^1_{(2)}(\mathcal{G})$. This is the same as saying that $\delta_j(f) = \rho_{g_j}(f) - f \in \ell^2(G)$ for each $j = 1,\dots,n$.

Denoting again by m_f the operator of multiplication by f, we have

$$\begin{aligned}[m_f, JU_jJ] &= m_fJU_jJ - JU_jJm_f = JU_jJ(JU_j^{-1}Jm_fJU_jJ - m_f)\\ &= JU_jJ(m_{\rho_{g_j}(f)} - m_f) = JU_jJm_{\delta_j(f)}\,. \end{aligned} \tag{3.7}$$

Since $\delta_j(f) \in \ell^2(G)$, we have that $m_{\delta_j(f)} \in HS$ and so also $[m_f, JU_jJ]\in HS$. Thus m_f is a bounded operator whose commutators with JU_jJ, $j = 1,\dots,n$, are Hilbert–Schmidt operators.

Thus

$$A = \left\{([m_f, JU_1J],\dots,[m_f, JU_nJ]) : f \in \ell^\infty(G) \text{ s.t. } \delta f \in C^1_{(2)}(\mathcal{G})\right\} \subset H_0(U_1,\dots,U_n)\,.$$

Since $H_0(U_1,\dots,U_n)$ is an M,M-bimodule, it will suffice to prove that

$$\dim_{M\bar{\otimes}M^o} \overline{M\ A\ M} \geq \delta^{(2)}(G)$$

(we'll actually prove that $\dim_{M\bar{\otimes}M^o} \overline{M\,A\,M} = \delta^{(2)}(G)$.)

We now aim to use Lemma 3.4 and the map ϕ defined there. Consider the M,M-bimodule isomorphism of HS^n given by

$$\Psi : (\Xi_1,\dots,\Xi_n) \mapsto (JU_1^{-1}J\Xi_1,\dots,JU_n^{-1}J\Xi_n)\,.$$

Then if $f \in \ell^\infty(G)$ with $\delta_j(f) \in \ell^2(G)$, $j = 1,\dots,n$, we have that

$$\Psi\big([m_f, JU_1J],\dots,[m_f, JU_nJ]\big) = (m_{\delta_1(f)},\dots,m_{\delta_n(f)}) = \phi\big(\delta_1(f),\dots,\delta_n(f)\big)\,.$$

Hence

$$\begin{aligned}\dim_{M\bar{\otimes}M^o} \overline{M\,A\,M} &= \dim_{M\bar{\otimes}M^o} \overline{\Psi(M\,A\,M)}\\ &= \dim_{M\bar{\otimes}M^o} \overline{M\phi\big(\{c \in C^1_{(2)}(\mathcal{G}) : c = \delta f \text{ for some } f \in \ell^\infty(G)\}\big)M}\\ &= \dim_M \overline{\left\{c \in C^1_{(2)}(\mathcal{G}) : c = \delta f \text{ for some } f \in \ell^\infty(G)\right\}} = \delta^{(2)}(M)\,,\end{aligned}$$

using Lemma 3.4 and Lemma 3.2 in the last two equalities. $\square$

For any algebra A generated by a self-adjoint set of operators $X_1,\dots,X_n$ on some Hilbert space H, and a tracial state on A given by $\tau(X) = \langle X\xi, \xi\rangle$, for some fixed $\xi \in H$, let

$$\Delta(X_1,\dots,X_n) = n - \dim_{M\bar{\otimes}M^o} \overline{\left\{(T_1,\dots,T_n) \in FR^n : \sum_j [T_j, JX_jJ] = 0\right\}}\,,$$

where $M = W^*(X_1, \dots, X_n)$ is the von Neumann algebra generated by A, FR stands for finite-rank operators on $L^2(W^*(X_1, \dots, X_n))$, and the closure is taken in the Hilbert–Schmidt norm. This quantity was introduced in [CoS] and is related to L^2-homology of A. The appearance of FR comes from the fact that this is the analogue of the space of compactly supported functions on the group, in the same way that HS is the analogue of the space of square-summable functions. One has:

$$\Delta(X_1, \dots, X_n) = \beta_1(X_1, \dots, X_n) - \beta_0(X_1, \dots, X_n) + 1$$

(we refer to [CoS] for a definition of these Betti numbers). By [CoS] one always has the inequality

$$\underline{\Delta}(X_1, \dots, X_n) \leq \Delta(X_1, \dots, X_n) .$$

We sketch the proof for completeness. Let $D \in B(L^2(M))$ be such that $S_j = [JX_jJ, D] \in HS$, $j = 1, \dots, n$. Then if $T_j \in FR$ satisfy

$$\sum_j [T_j, JX_jJ] = 0 ,$$

we have

$$0 = \mathrm{Tr}\Big(\sum_j [T_j, JX_jJ]^* D\Big) = \sum_j \mathrm{Tr}\left(T_j^*[JX_jJ, D]\right) = \sum_j \mathrm{Tr}(T_j^* S_j) .$$

Thus $(T_1, \dots, T_n) \perp (S_1, \dots, S_n)$ in HS^n. Hence

$$H_0(X_1, \dots, X_n) \perp \Big\{(T_1, \dots, T_n) \in FR^n : \sum_j [T_j, JX_jJ] = 0\Big\} .$$

Since the Murray–von Neumann dimension of HS^n over $M \bar{\otimes} M^o$ is n, it follows that $\underline{\Delta} \leq \Delta$.

COROLLARY 3.6. *Let $Y_1, \dots, Y_n$ be a self-adjoint set of generators of $\mathbb{C}G$. Then*

$$\Delta(Y_1, \dots, Y_n) = \underline{\Delta}(Y_1, \dots, Y_n) = \delta^{(2)}(G) ,$$

where $\delta^{(2)}(G) = \beta_1(G) - \beta_0(G) + 1$.

Proof. Since both Δ and $\underline{\Delta}$ don't depend on the choice of generators of $\mathbb{C}G$, we may as well assume that $Y_1 = U_1, \dots, Y_n = U_n$ correspond to a symmetric family of generators of G. We then have by [CoS, Theorem 3.3(c)] and Lemma 3.5 that

$$\delta^{(2)}(G) \geq \Delta(U_1, \dots, U_n) \geq \underline{\Delta}(U_1, \dots, U_n) \geq \delta^{(2)}(G) ,$$

which forces all inequalities to be equalities. □

4 Computation of Free Entropy Dimension

Let G be a finitely generated discrete group, and choose $Y_1, \ldots, Y_n \in \mathbb{C}G$ to be self-adjoint elements in group algebra of G that generate it as a complex algebra. One could for example take $Y_{2j} = \operatorname{Re} \lambda_{g_j} = \frac{1}{2}(\lambda_{g_j} + \lambda_{g_j}^{-1})$, $Y_{2j-1} = \operatorname{Im} \lambda_{g_j} = \frac{1}{2i}(\lambda_{g_j} - \lambda_{g_j}^{-1})$, $j = 1, \ldots, n$, for some generators $g_1, \ldots, g_n$ of G.

Theorem 4.1. *Let G be a finitely generated group. Let $Y_1, \ldots, Y_n$ be any self-adjoint generators of the group algebra $\mathbb{C}G$, equipped with the canonical group trace τ. Then*

$$\delta^*(Y_1, \ldots, Y_n) = \delta^\star(Y_1, \ldots, Y_n) = \beta_1(G) - \beta_0(G) + 1\,.$$

In particular, δ^* is an invariant of the algebra generated by $Y_1, \ldots, Y_n$, taken with its trace.

Proof. By [S, Corollary 2.12]

$$\delta^*(Y_1, \ldots, Y_n) \geq \underline{\Delta}(Y_1, \ldots, Y_n)\,. \tag{4.1}$$

By [CoS, Theorem 4.4 and Corollary 4.6],

$$\Delta(Y_1, \ldots, Y_n) \geq \delta^\star(Y_1, \ldots, Y_n) \geq \delta^*(Y_1, \ldots, Y_n). \tag{4.2}$$

Combining (4.1), (4.2) and Corollary 3.6, we find that

$$\begin{aligned}\delta^{(2)}(G) = \Delta(Y_1, \ldots, Y_n) &\geq \delta^\star(Y_1, \ldots, Y_n) \\ &\geq \delta^*(Y_1, \ldots, Y_n) \geq \underline{\Delta}(Y_1, \ldots, Y_n) = \delta^{(2)}(G)\,,\end{aligned}$$

as claimed. □

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