ℓ_1-HOMOLOGY OF COMBABLE GROUPS AND 3-MANIFOLD GROUPS.

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ABSTRACT. S. Gersten asked whether the *reduced* ℓ_1-homology of a group of type $\mathcal{F}_n$ vanishes in all dimensions up to $n-1$. We prove this for combable groups and for the fundamental groups of closed 3-manifolds. This means that the ℓ_1-homology in these cases can be interpreted as "an amount of non-linearity" for the isoperimetric functions.

0. Introduction.

Exotic homology theories for groups developed by M. Gromov and S. Gersten, the ℓ_1-homology $H_*^{(1)}$ in particular, proved to be useful tools for describing geometric properties of groups. For example, hyperbolic groups are characterized by vanishing of certain reduced and non-reduced ℓ_1-homology (Allcock-Gersten [1]). Later improving that result Gersten showed that a finitely presented group is hyperbolic iff $H_1^{(1)}(G,\mathbb{R}) = 0$. Also, he gave a homological description of when embeddings of finitely generated groups are undistorted in the word metric.

Let G be a group of type $\mathcal{F}_n$, i.e. there is a cell complex X' of type $K(G,1)$ with finite n-skeleton. Let X be the universal covering of X'. The present paper is an attempt to answer the following

Question (Gersten). *For $i \leq n-1$, can every summable cellular (real-valued) i-cycle in X be approximated in ℓ_1-norm by i-cycles of compact support? Equivalently, does the reduced ℓ_1-homology $\bar{H}_i^{(1)}(G,\mathbb{R})$ vanish? (See the definitions below.)*

Allcock and Gersten [1] showed the affirmative answer for $i = 1$. In this paper we prove the affirmative answer for all $i \geq 1$ when G is a combable group or the fundamental group of a closed 3-manifold. Moreover, for combable groups this holds for chains with arbitrary coefficients.

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1. ℓ_1-HOMOLOGY.

In this section we remind some definitions and results from Gersten's theory that will explain why the above question is interesting. We consider the general setting with arbitrary abelian coefficients.

1.1. **Normed abelian groups.** Let A be an abelian group. An *abelian group norm* on A is a function $|\cdot|: A \to \mathbb{R}_+$ satisfying (1) $|a| = 0$ iff $a = 0$, (2) $|-a| = |a|$, and (3) $|a + a'| \leq |a| + |a'|$ for all $a, a' \in A$. An abelian group with an abelian group norm is called a *normed abelian group*. In particular, each normed abelian group is a metric space.

1.2. **Homology theories.** Let G be of type $\mathcal{F}_n$, so that there exists a cellular $K(G,1)$ complex with finite n-skeleton. Let X be the universal covering of this $K(G,1)$ complex and A be a normed abelian group.

Denote $C_i^{(1)}(X, A)$ the set of the cellular (not necessarily compactly supported) A-valued i-chains of finite ℓ_1-norm $|\cdot|_1$, where

$$|c|_1 := \sum_e |c(e)|$$

and e runs over the i-cells of X. The ℓ_1-norm makes $C_i^{(1)}(X, A)$ a normed abelian group. The elements of $C_i^{(1)}(X, A)$ are called *summable chains*.

Define the boundary homomorphism $\hat{\partial}_i : C_i^{(1)}(X, A) \to C_{i-1}^{(1)}(X, A)$ in the obvious way, as the extension of the usual boundary homomorphism. It is well-defined for $i \leq n$ because the n-skeleton of X is finite modulo G-action. For the same reason, there exists a constant K_i depending only on i and a choice of X, such that $|\hat{\partial}_i c|_1 \leq K_i\, |c|_1$, i.e. $\hat{\partial}_i$ is bounded, and therefore, continuous.

The *ℓ_1-homology* of G is the homology of the chain complex

$$\ldots \xrightarrow{\hat{\partial}} C_2^{(1)}(X, A) \xrightarrow{\hat{\partial}} C_1^{(1)}(X, A) \xrightarrow{\hat{\partial}} C_0^{(1)}(X, A) \xrightarrow{\hat{\partial}} 0,$$

i.e.

$$H_i^{(1)}(G, A) := Z_i^{(1)}(X, A)/B_i^{(1)}(X, A), \quad i \leq n-1,$$

where

$$Z_i^{(1)}(X, A) := Ker\ \hat{\partial}_i \quad \text{and} \quad B_i^{(1)}(X, A) := Im\ \hat{\partial}_{i+1}.$$

$H_i^{(1)}(G, A)$ is well defined in dimensions $i \leq n-1$.

The *reduced ℓ_1-homology* of G is

$$\bar{H}_i^{(1)}(G, A) := Z_i^{(1)}(X, A)/\bar{B}_i^{(1)}(X, A), \quad i \leq n-1,$$

where $\bar{B}_i^{(1)}(X, A)$ is the closure of $B_i^{(1)}(X, A)$ in $C_i^{(1)}(X, A)$.

Note that both $H_i^{(1)}(G, A)$ and $\bar{H}_i^{(1)}(G, A)$ are defined only when G is of type $\mathcal{F}_{i+1}$. Always when talking about $H_i^{(1)}(G, A)$ or $\bar{H}_i^{(1)}(G, A)$ in this paper we assume by default that G is of type $\mathcal{F}_{i+1}$.

1.3. **Independence of X.** The ℓ_1-homology and the reduced ℓ_1-homology are quasiisometry invariant. This can be shown similarly to quasiisometry invariance of ℓ_∞-cohomology in [6]. In particular, $\bar{H}_i^{(1)}(G,A)$ and $H_i^{(1)}(G,A)$ do not depend on the choice of a $K(G,1)$ complex.

1.4. **Gersten's short exact sequence.** Denote by $Z_i(X,A)$ and $B_i(X,A)$ the sets of usual compactly supported i-cycles and i-boundaries in X, respectively, and by $\bar{Z}_i(X,A)$ and $\bar{B}_i(X,A)$ their closures in $C_i^{(1)}(X,A)$. Obviously,

$$B_i(X,A) \subseteq B_i^{(1)}(X,A). \tag{1}$$

Each element of $B_i^{(1)}(X,A)$ has form $\hat{\partial}c$, where c is a summable chain. Since c can be approximated by compactly supported chains and $\hat{\partial}$ is bounded, then $\hat{\partial}c$ can be approximated by boundaries of compactly supported chains. This implies that

$$B_i^{(1)}(X,A) \subseteq \bar{B}_i(X,A). \tag{2}$$

By (1) and (2),

$$\bar{B}_i^{(1)}(X,A) = \bar{B}_i(X,A). \tag{3}$$

Since X is contractible, $B_i(X,A) = Z_i(X,A)$ for $i \geq 1$, and their closures in $C_i^{(1)}(X,A)$ also coincide:

$$\bar{B}_i(X,A) = \bar{Z}_i(X,A). \tag{4}$$

Obviously, $Z_i(X,A) \subseteq Z_i^{(1)}(X,A)$ and $Z_i^{(1)}(X,A)$ is closed in $C_i^{(1)}(X,A)$ as the kernel of $\hat{\partial}_i$, hence

$$\bar{Z}_i(X,A) \subseteq Z_i^{(1)}(X,A) \tag{5}$$

Combining (2), (3), (4) and (5), we obtain the filtration

$$B_i^{(1)}(X,A) \subseteq \bar{B}_i^{(1)}(X,A) \subseteq Z_i^{(1)}(X,A),$$

which together with the definitions of $H_i^{(1)}$ and $\bar{H}_i^{(1)}$ yields the short exact sequence

$$0 \to \bar{B}_i^{(1)}(X,A)/B_i^{(1)}(X,A) \to H_i^{(1)}(G,A) \to \bar{H}_i^{(1)}(G,A) \to 0. \tag{6}$$

The kernel and the cokernel of the above short exact sequence should be viewed as "parts" of $H_i^{(1)}(G,A)$. We take a closer look at these parts now.

By (3) and (4), $\bar{Z}_i(X,A) = \bar{B}_i^{(1)}(X,A)$, hence

$$\bar{H}_i^{(1)}(G,A) = Z_i^{(1)}(X,A)/\bar{Z}_i(X,A) \tag{7}$$

Therefore, vanishing of $\bar{H}_i^{(1)}(G,A)$ means exactly that summable i-cycles in X can be approximated in ℓ_1-norm by compactly supported i-cycles.

Let G, X and A be as above. We say that G satisfies *a linear isoperimetric inequality* for A-valued i-cycles if there exists a constant $K \geq 0$ such that for any cycle $z \in Z_i(X,A)$

there exists an $(i+1)$-chain $a \in C_i(X, A)$ with $\partial a = z$ and $|a|_1 \le K|z|_1$. Similarly to [6] one can see that this property is quasiisometry invariant, therefore independent of the choice of X.

Proposition 1. *If A is complete and G satisfies a linear isoperimetric inequality for A-valued i-cycles then $\bar{B}_i^{(1)}(X, A)/B_i^{(1)}(X, A) = 0$.*

Proof. The abelian group $B_i(X, A)$ has the ℓ_1-norm $|\cdot|_1$ induced from $C_i(X, A)$. It can also be given the filling norm $|\cdot|_f$ defined by

$$|b|_f := \inf \left\{ \, |a|_1 \;\middle|\; a \in C_{i+1}(X, A) \text{ and } \partial a = b \, \right\}.$$

Since the boundary map ∂ is bounded, $|\cdot|_1$ is dominated by $|\cdot|_f$. If G satisfies a linear isoperimetric inequality for A-valued i-cycles, then these norms are actually equivalent.

Since $C_{i+1}^{(1)}(X, A)$ is the completion of $C_{i+1}(X, A)$, then $B_i^{(1)}(X, A)$ is the completion of $B_i(X, A)$ in the filling norm. Since $C_i^{(1)}(X, A)$ is complete, then $\bar{B}_i^{(1)}(X, A) = \bar{B}_i(X, A)$ is the completion of $B_i(X, A)$ in the ℓ_1-norm. Since the two norms on $B_i(X, A)$ are equivalent, the two completions coincide, i.e. $\bar{B}_i^{(1)}(X, A) = B_i^{(1)}(X, A)$. □

Proposition 1 says that $\bar{B}_i^{(1)}(X, A)/B_i^{(1)}(X, A)$ is a measure of non-linearity for isoperimetric functions in dimension i.

2. Approximation of summable cycles.

Allcock and Gersten [1] asked the following: if X is a simply connected 2-complex, can summable 2-cycles on X be approximated by 2-cycles of compact support? The example below shows that the answer is "no" even assuming that the complex is contractible.

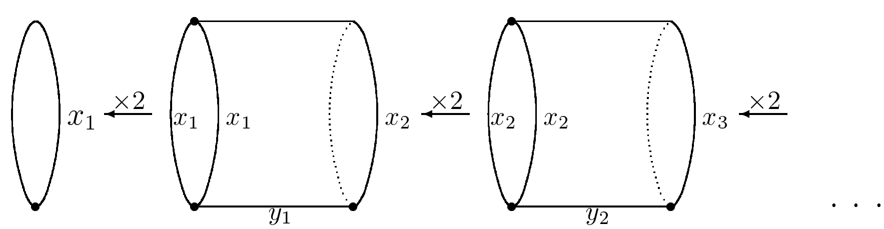

Figure 1. 2-complex Y.

The example. Start with a 2-cell f_1 whose boundary consists of one-edge loop labeled x_1. Take another 2-cell f_2 with the boundary labeled $y_1 x_2 y_1^{-1} x_1^{-2}$. Glue these cells identifying the edges with the same label to obtain a 2-complex Y_1. Take a 2-cell f_3 with boundary labeled $y_2 x_3 y_2^{-1} x_2^{-2}$, glue it to Y_1 identifying corresponding edges to obtain a complex Y_2. Repeat this procedure infinitely many times. This gives an infinite 2-complex Y(see Fig. 1). Each complex Y_i is easily seen to be contractible, hence so is Y, as a union of contractible spaces.

The assignment $z(f_i) := \frac{1}{2^i}$ gives a summable 2-chain on Y, and z is a cycle since $(\partial z)(x_i) = (\partial f_i)(x_i) - (\partial f_{i+1})(x_i) = \frac{1}{2^i} - 2\frac{1}{2^{i+1}} = 0$ and $(\partial z)(y_i) = 0$. So z is a summable 2-cycle on Y, and z is not approximable by cycles of compact support since all such are trivial on Y.

Unfortunately, the above contractible complex does not admit a cocompact group action. In [1] there is an example due to E. Formanek of a (not simply connected) complex X with $Z_2(X, \mathbb{R}) = 0$ and $Z_2^{(1)}(X, \mathbb{R}) \neq 0$. These examples suggest that the approximation of summable cycles might not be possible in general.

Note that, for any (finitely generated) group G, $\bar{H}_0^{(1)}(G, \mathbb{R}) = 0$, since any summable function on a countable set can be approximated by functions of finite support. Also, $\bar{H}_1^{(1)}(G, \mathbb{R}) = 0$ for any G, as shown in [1]. In general, the question of vanishing of the ℓ_1-homology for groups of type $\mathcal{F}_n$ remains open.

3. Combings.

Definition 2. *A metric space (X, d) with a base point $*$ is called* <u>combable</u> *if it admits a* bounded combing $\{p_v \mid v \in X\}$, *i.e. a set with the following properties:*

1. *For each $v \in X$, $p_v : [0, \infty) \to X$ is a (not necessarily continuous) map with $p_v(0) = *$ and $p([t_v, \infty)) = v$ for some $t_v \in [0, \infty)$.*
2. *The restriction $p_v|_{[0,t_v]}$ is a (k, λ)-geodesic, i.e.*
$$\frac{1}{k} d(p_v(t), p_v(t')) - \lambda \leq |t' - t| \leq k d(p_v(t), p_v(t')) + \lambda$$
for any $t, t' \in [0, t_v]$ and some $k \geq 1$, $\lambda \geq 0$ independent of v,t and t'.
3. *$d(p_v(t), p_w(t)) \leq A\, d(v, w) + B$ for some $A > 0$ and $B > 0$ independent of v, w and t.*

A finitely generated group G is called <u>combable</u> *if it is combable as a metric space with the word metric with respect to some finite generating set.*

J. Alonso and M. Bridson proved the following

Theorem 3 ([2],Theorem 1.1)**.** *The property of being combable is quasiisometry invariant.*

and the following was shown by M. Troyanov.

Theorem 4 ([9],Proposition 19)**.** *Let X be a proper geodesic space and let G act properly discontinuously and cocompactly by isometries on X. Then G is finitely generated and G with the word metric (with respect to a finite generating set) is quasiisometric to X.*

The main result of this section is

Theorem 5. *If G is a combable group and A is a normed abelian group, then $\bar{H}_n^{(1)}(G, A) = 0$ for any $n \geq 1$.*

Proof. Since G is combable, it possesses a $K(G,1)$ complex X' with finitely many cells in each dimension [3, 5], so $\bar{H}_n^{(1)}(G,\mathbb{R})$ is defined for any n. Let X be the universal covering of X' and let a vertex $*$ in X be the base point.

Define d to be the path metric on $X^{(1)}$ induced by assigning length 1 to each edge. Put the induced metric on $X^{(0)}$ and put a word metric on G (with respect to some finite generating set). Since G acts on $X^{(1)}$ freely and cocompactly, each inclusion in the sequence $X^{(0)} \hookrightarrow X^{(1)} \hookleftarrow G$ is a quasiisometry. Hence, by Theorem 3, $X^{(0)}$ is combable. Let $\{p_v \mid v \in X^{(0)}\}$ be a bounded combing on $X^{(0)}$.

Denote $B(R)$ the union of all cells in $X^{(n+1)}$ whose vertices lie at distance at most R from $*$. Define a projection $pr^R : X^{(0)} \to X^{(0)}$ by the rule $pr^R(v) := p_v(R)$. Conditions (1) and (2) in Definition 2 of a combable space imply the following.

Lemma 6. (a) *The image of $X^{(0)}$ under pr^R lies in $B(kR+\lambda)^{(0)}$.*
(b) *pr^R fixes the vertices of $B(\frac{R}{k}-\lambda)$.*

The map pr_R induces a map $pr_0^R : C_0^{(1)}(X,A) \to C_0(X,A)$.

Theorem 7. *The map $pr_0^R : C_0^{(1)}(X,A) \to C_0(X,A)$ extends to a chain map $pr_*^R : C_*^{(1)}(X,A) \to C_*(X,A)$ with the following properties:*

(a) *there exists a sequence of numbers $K_i \geq 1$ such that $|pr_i^R(c)|_1 \leq |c|_1$ for all $c \in C_i^{(1)}(X,A)$,*
(c) *if $supp(c) \subseteq B(\frac{R}{k}-\lambda)$, then $pr_i^R(c) = c$, and*
(b) *there exists a sequence of numbers $\delta_i \geq 0$ such that $supp(pr_i^R(c)) \subseteq B(kR+\lambda+\delta_i)$ for all $c \in C_i^{(1)}(X,A)$.*

Proof. For $i=0$, conditions $(a)-(c)$ hold by Lemma 6. Each edge e with endpoints v,w we map onto a shortest edge path γ in $X^{(1)}$ connecting $pr^R(v)$ to $pr^R(w)$. We view γ as an integral 1-chain a_e with $\partial a_e = w - v$. By property (3) in the definition of a bounded combing,

$$|a_e|_1 = length(\gamma) = d(pr_0^R(v), pr_0^R(w)) \leq k\, d(v,w) + \lambda = k + \lambda.$$

Up to the G-action, there are only finitely many pairs of vertices at distance at most $k+\lambda$ from each other, hence we can choose a 1-chain a_e so that the image of the map $e \mapsto a_e$ is finite up to G-action, and moreover, (b) is satisfied. In particular, $|a_e|_1$ is uniformly bounded over all edges e, so the formula

$$pr_1^R\Big(\sum_e \alpha_e e\Big) := \sum_e \alpha_e a_e, \quad \alpha_k \in A$$

defines a bounded map $pr_1^R : C_1^{(1)}(X,A) \to C_1^{(1)}(X,A)$.

We take δ_i to be the maximum over all edges e of the diameters of $supp\, a_e$, then (c) is satisfied. By the definition of pr_1^R, the following diagram commutes:

$$\begin{array}{ccc} C_1^{(1)}(X,A) & \xrightarrow{\partial} & C_0^{(1)}(X,A) \\ \downarrow pr_1^R & & \downarrow pr_0^R \\ C_1(X,A) & \xrightarrow{\partial} & C_0(X,A) \end{array}$$

Next we continue inductively constructing pr_i^R the same way as we constructed pr_1^R. □

Now we finish the proof of Theorem 5. Let z be a summable cycle of dimension i in X. For each R, $z = z_R + z'_R$, where z_R is a chain supported on $B(\frac{R}{k} - \lambda)$, and z'_R is a chain supported on the cells not in $B(\frac{R}{k} - \lambda)$. Given $\epsilon > 0$, we choose $R = R(z, \epsilon, k, \lambda)$ large enough so that $|z'_R|_1 \leq \frac{\epsilon}{2K_i}$. Then by Theorem 7,

$$\left|z - pr_i^R(z)\right|_1 = \left|z'_R - pr_i^R(z'_R)\right|_1 \leq \left|z'_R\right|_1 + \left|pr_i^R(z'_R)\right|_1 \leq K_i\frac{\epsilon}{2K_i} + \frac{\epsilon}{2K_i} \leq \epsilon.$$

Since pr_*^R is a chain map, $pr_i^R(z)$ is a compactly supported cycle, and it is ϵ-close to z by the above formula. □

There are two immediate corollaries of Theorem 5.

Theorem 8. *If G is an automatic group, then $\bar{H}_n^{(1)}(G, \mathbb{R}) = 0$ for any $n \geq 1$.*

Proof. Theorems 2.3.9 and 2.5.1 in [5] imply that the Cayley graph of G with respect to a finite generating set admits a bounded combing. □

Theorem 9. *If G is the fundamental group of a finite non-positively curved complex, then $\bar{H}_n^{(1)}(G, \mathbb{R}) = 0$ for any n.*

"Non-positively curved" here means that there is $K \leq 0$ so that

(1) each simplex of the complex is isometric to the convex hull of finitely many points in the standard space of constant curvature K,
(2) inclusions of faces are isometric embeddings, and
(3) $CAT(1)$ condition is satisfied on the links of the vertices.

Proof of Theorem 9. By a result of W. Ballmann [4] the universal cover X of such a complex is $CAT(0)$, hence contractible. It also implies that the set of geodesic paths connecting the base point $*$ to the points in X gives a bounded combing in X, hence G is combable. □

4. Graphs.

In this section we prove some preliminary results for graphs which will be used in section 5 for 3-manifolds.

Let Γ be a locally finite graph. Fix a base point $*$ in $\Gamma^{(0)}$. We will view each edge of Γ as an oriented interval of length 1 with the initial and terminal endpoints $i(e)$ and $t(e)$, respectively; $\bar{e}$ will denote the edge e with the opposite orientation. Γ is a metric space with the metric given by the length of a shortest path. We will need an extended notion of a subgraph.

Definition 10. *A subset U of Γ is called a* <u>subgraph</u> *of Γ if, for each closed edge e in Γ, $U \cap e$ is an interval (possibly degenerate or empty) equipped with an orientation. For a subgraph U, $|U|$ denotes the set U with the Lebesgue measure on it.*

The orientation on $U \cap e$ does not have to be induced by the orientation on e. Obviously, the closure of U is a genuine graph with an orientation and a length assigned to each of its edges.

Let c be a real valued 1-coboundary on Γ. In other words, c is a function $c : E(\Gamma) \to \mathbb{R}$ on the set of directed edges in Γ so that $c(e) = -c(\bar{e})$, and with the additional property that $c = \delta P$ for some 0-cochain P which can be thought of as a "potential function" $P : \Gamma^{(0)} \to \mathbb{R}$ and it can be constructed as follows: for each $v \in \Gamma^{(0)}$ choose an edge path p_v in Γ connecting $*$ to v, and let $P(v)$ be the sum of values of c on the directed edges in p_v. Since c is a coboundary the sum of its values along any directed edge circuit is 0, so the above definition does not depend on the choice of p_v for v. One easily checks that $c = \delta P$.

Now we extend P linearly to a function $\bar{P}$ on all of Γ: for a point $(1-x)i(e) + xt(e)$ on an edge e, $0 \le x \le 1$, define

$$\begin{aligned} &\bar{P}((1-x)i(e) + xt(e)) := (1-x)P(i(e)) + xP(t(e)) = \\ &P(i(e)) + x[P(t(e)) - P(i(e))] = P(i(e)) + xc(e). \end{aligned} \tag{8}$$

The interior of each edge e in Γ has the standard basis $\dfrac{\partial}{\partial x}$ canonically determined by the orientation on e. The restriction of the potential $\bar{P}$ to $\Gamma \setminus \Gamma^{(0)}$ can be viewed as a 0-form. We denote φ_c the differential of $\bar{P}$, i.e. $\varphi_c := \dfrac{\partial \bar{P}}{\partial x} dx$ is a 1-form on $\Gamma \setminus \Gamma^{(0)}$ and (8) says that, when restricted to an edge e, φ_c is a constant form equal to $c(e)dx$. Actually, φ_c can be defined for any 1-cochain c this way: $c(e)dx$ on each edge e. If $U \subseteq \Gamma$ is a subgraph, then it makes sense to integrate φ_c over U defining

$$\int_U \varphi_c := \sum_{e \in E(\Gamma)} \int_{U \cap int(e)} \varphi_c.$$

Here $U \cap int(e)$ is given the orientation with respect to U. The following lemma is just the fundamental theorem of calculus for a piecewise constant function.

Lemma 11. *Let c be a 1-coboundary on Γ. If γ is a piecewise linear path going from a point x to a point y in Γ, then $\int_\gamma \varphi_c = \bar{P}(y) - \bar{P}(x)$. In particular, $\int_\gamma \varphi_c$ does not depend on the choice of γ.*

Lemma 12. *Let c be a 1-cochain on Γ. Then*

(a) $\left| \int_U \varphi_c \right| \le \int_{|U|} |\varphi_c|$, *for any subgraph U of Γ, and*

(b) $|c|_1 = \int_{|\Gamma|} |\varphi_c|$, *where $|\cdot|_1$ is the ℓ_1-norm on $C_1(\Gamma, \mathbb{R})$ with respect to the standard basis.*

Proof. ***(a)***

$$\left|\int_U \varphi_c\right| = \left|\sum_{e\in E(\Gamma)} \int_{U\cap int(e)} \varphi_c\right| \leq \sum_{e\in E(\Gamma)} \left|\int_{U\cap int(e)} \varphi_c\right| \leq$$
$$\sum_{e\in E(\Gamma)} \int_{|U\cap int(e)|} |\varphi_c| = \int_{|U\setminus\Gamma^{(0)}|} |\varphi_c| = \int_{|U|} |\varphi_c|.$$

The last inequality holds because $\Gamma^{(0)}$ is of Lebesgue measure 0.
(b)

$$|c|_1 = \sum_{e\in E(\Gamma)} |c(e)| = \sum_{e\in E(\Gamma)} \int_{|e|} |\varphi_c| = \int_{|\Gamma|} |\varphi_c|.$$

Lemma 12 is proved. □

Definition 13. *A connected locally finite graph Γ is called* one-ended *if for any bounded set $K \subseteq \Gamma$ there exists a bounded set $K' \subseteq \Gamma$ so that $K \subseteq K'$ and $\Gamma \setminus K'$ is nonempty and connected. A finitely generated group is called* one-ended *if its Cayley graph with respect to a finite generating set is one-ended.*

Let $B^1_c(\Gamma,\mathbb{R})$ be the space of all 1-coboundaries of compact support, and $B^1_{sum}(\Gamma,\mathbb{R})$ be the space of all summable 1-coboundaries on Γ, i.e. $B^1_{sum}(\Gamma,\mathbb{R}) = B^1(\Gamma,\mathbb{R}) \cap C^1_{(1)}(\Gamma,\mathbb{R})$, equipped with the ℓ_1-norm. Obviously, $B^1_c(\Gamma,\mathbb{R}) \subseteq B^1_{sum}(\Gamma,\mathbb{R})$. We denote by $\bar{B}^1_c(\Gamma,\mathbb{R})$ the closure of $B^1_c(\Gamma,\mathbb{R})$ in $C^1_{(1)}(\Gamma,\mathbb{R})$, the space of summable 1-cochains.

Proposition 14. *If Γ is a one-ended graph then $B^1_{sum}(\Gamma,\mathbb{R}) \subseteq \bar{B}^1_c(\Gamma,\mathbb{R})$.*

Proof. We need to show that any summable 1-coboundary on Γ can be approximated in ℓ_1-norm by 1-coboundaries of compact support.

Since Γ is one-ended, it must be unbounded. Let c be a summable 1-coboundary on Γ and $\bar{P}$ be its potential function with respect to the base point $*$. $\bar{P}$ may be thought of as a function $\Gamma \to [-\infty,\infty]$, where $[-\infty,\infty]$ is the two-point compactification of $\mathbb{R}$. For $t \in [-\infty,\infty]$ define $U_t := \bar{P}^{-1}([-\infty,t])$ and $V_t := \bar{P}^{-1}([t,\infty])$ (see Fig.2). In particular, $U_{-\infty} = \emptyset$ and $V_\infty = \emptyset$ because $\bar{P}(z)$ is always a real number. Let

$$u := \{t \in [-\infty,\infty] \mid U_t \textit{ is bounded}\},$$
$$v := \{t \in [-\infty,\infty] \mid V_t \textit{ is bounded}\}.$$

Lemma 15. *Let Γ be an connected, infinite, locally finite graph, and u and v be as above. Then the following properties are satisfied.*

(a) *$-\infty \in u$ and $\infty \in v$.*
(b) *The sets u and v are disjoint non-degenerate subintervals of $[-\infty,\infty]$.*
(c) *If Γ is one-ended, then $\sup u = \inf v \in (-\infty,\infty)$.*

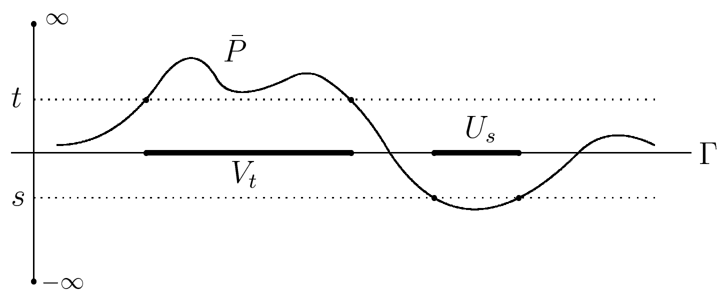

FIGURE 2. The sets U_s and V_t.

Proof. ***(a)*** The sets $U_{-\infty}$ and V_∞ are empty and hence bounded.

(b) If there were $t \in u \cap v$, it would imply that $\Gamma = U_t \cup V_t$ is bounded, therefore Γ is a finite graph, which is not the case. So u and v are disjoint.

If $t \in u$ then U_t is bounded and, for any s with $-\infty \le s \le t$, $U_s \subseteq U_t$ is bounded. It means that u is a subinterval of $[-\infty, \infty]$ containing $-\infty$. Moreover, u is a non-degenerate interval, because $|\bar{P}|$ is bounded by the ℓ_1-norm of c. Analogously, v is a non-degenerate subinterval of $[-\infty, \infty]$ containing ∞.

(c) Denote $a := \sup u$, $b := \inf v$. As we just saw, a and b are real numbers. Since u and v are disjoint, $a \le b$. Suppose $a < b$. Pick some a' and b' so that $a < a' < b' < b$. Then the sets $\bar{P}^{-1}([-\infty, a'))$ and $\bar{P}^{-1}((a', \infty])$ are unbounded and $\bar{P}^{-1}(a')$ separates them from each other. Hence $\bar{P}^{-1}(a')$ is unbounded, since Γ would not be one-ended otherwise. Analogously, $\bar{P}^{-1}(b')$ is unbounded.

Now we choose arbitrary points x_1 and y_1 in $\bar{P}^{-1}(a')$ and $\bar{P}^{-1}(b')$, respectively, and connect x_1 to y_1 by a piecewise linear path p_1. It is possible to do because $\bar{P}^{-1}(a')$ and $\bar{P}^{-1}(b')$ are non-empty and Γ is connected. By removing loops we can assume that p_1 is injective. Since Γ is one-ended, there is a bounded set K_1 containing (the image of) p_1 so that $\Gamma \setminus K_1$ is connected. Since $\bar{P}^{-1}(a')$ and $\bar{P}^{-1}(b')$ are unbounded, we can choose points x_2 and y_2 in $\bar{P}^{-1}(a') \setminus K_1$ and $\bar{P}^{-1}(b') \setminus K_1$, respectively, and connect x_2 to y_2 by an injective path p_2 in $\Gamma \setminus K_1$. Choose a bounded K_2 containing p_1 and p_2, and so on. In this way we obtain an infinite collection of disjoint paths p_i connecting x_i to y_i, where $\bar{P}(x_i) = a'$ and $\bar{P}(y_i) = b'$. Since p_i's are injective, they can be viewed as subgraphs of Γ. Then, using lemmas 12 and 11,

$$\begin{aligned} |c|_1 = \; & \int_\Gamma |\varphi_c| \ge \int_{\cup_i |p_i|} |\varphi_c| = \sum_i \int_{|p_i|} |\varphi_c| \ge \sum_i \left| \int_{p_i} \varphi_c \right| \ge \\ & \ge \sum_i |\bar{P}(y_i) - \bar{P}(x_i)| = \sum_i |b' - a'| = \infty. \end{aligned}$$

This contradicts the assumption that c is summable. Lemma 15 is proved. □

We have $a = \sup u = \inf v \in (-\infty, \infty)$. Pick an increasing sequence $-\infty = s_0 < s_1 < s_2 < \dots$ in u and a decreasing sequence $\infty = t_0 > t_1 > t_2 > \dots$ in v so that

$\lim_{k\to\infty} s_k = a = \lim_{k\to\infty} t_k$. Let $W_k := U_{s_k} \cup V_{t_k}$ and define the functions $\psi_k : \Gamma \to \mathbb{R}$ to be equal to φ_c on W_k and 0 on $\Gamma \setminus W_k$. Note that W_k is a subgraph and it is bounded because $s_k \in u$ and $t_k \in u$.

Lemma 16. *For any directed edge loop γ in Γ, $\int_\gamma \psi_k = 0$.*

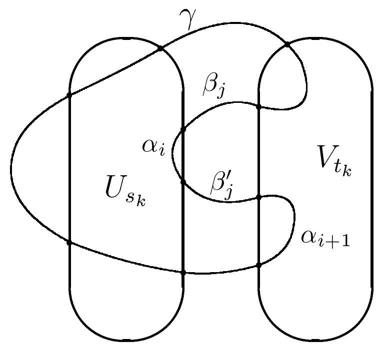

FIGURE 3. The subdivision of γ.

Proof. The loop γ can be subdivided as a concatenation of several piecewise linear paths: paths $\{\alpha_i\}$ with the images in $W_k = U_{s_k} \cup V_{t_k}$, paths $\{\beta_j\}$ with the images in $\Gamma \setminus W_k$ going from ∂U_{s_k} to ∂V_{t_k}, and paths $\{\beta'_j\}$ with the images in $\Gamma \setminus W_k$ going from ∂V_{t_k} to ∂U_{s_k} (see Fig. 3). Note that $\#\{\beta_j\} = \#\{\beta'_j\}$ because U_{s_k} and V_{t_k} are disjoint. The potential of ∂U_{s_k} is s_k and the potential of ∂V_{t_k} is t_k. Lemma 11 says that the integral of φ_c along any edge loop is 0, so

$$0 = \int_\gamma \varphi_c = \sum_i \int_{\alpha_i} \varphi_c + \sum_j \int_{\beta_j} \varphi_c + \sum_j \int_{\beta'_j} \varphi_c =$$

$$\sum_i \int_{\alpha_i} \psi_k + \sum_j (t_k - s_k) + \sum_j (s_k - t_k) = \sum_i \int_{\alpha_i} \psi_k = \int_\gamma \psi_k.$$

The last equality holds because ψ_k vanishes outside $U_{s_k} \cup V_{t_k}$. Lemma 16 is proved. ☐

For each k define a 1-cochain c_k on Γ by the rule $c_k(e) := \int_{int(e)} \psi_k$ and extending by linearity. Then, for any edge loop γ in Γ, by Lemma 16,

$$c_k(\gamma) = \sum_{e\in\gamma} c(e) = \sum_{e\in\gamma} \int_e \psi_k = \int_\gamma \psi_k = 0.$$

In other words, c_k vanishes on 1-cycles. Hence it admits a potential, i.e. c_k is a 1-coboundary. The value $c_k(e)$ is not 0 only for the edges e intersecting the support W_k of ψ_k. There are only finitely many such, so c_k has compact support. To finish the proof of Proposition 14 it only remains to show the following.

Lemma 17. *The sequence c_k converges to c in the ℓ_1-norm.*

Proof.

$$|c-c_k|_1 = \sum_{e\in E(\Gamma)} |c(e)-c_k(e)| = \sum_{e\in E(\Gamma)} \left|\int_{int(e)} (\varphi_c - \psi_k)\right| =$$

because φ_c and ψ_k coincide on W_k and ψ_k is 0 on $\Gamma \setminus W_k$,

$$(9) \qquad = \sum_{e\in E(\Gamma)} \left|\int_{int(e)\setminus W_k} \varphi_c\right| \le \sum_{e\in E(\Gamma)} \int_{|int(e)\setminus W_k|} |\varphi_c| = \int_{|\Gamma\setminus W_k|} |\varphi_c| = \sum_{l=k}^{\infty} \int_{|W_{l+1}\setminus W_l|} |\varphi_c|.$$

(Note that the domains of integration are subgraphs of Γ, so the integrals make sense.) Also, $|\Gamma| = \bigcup_{l=0}^{\infty} |W_l| \bigcup \bar{P}^{-1}(a) = \bigcup_{l=0}^{\infty} |W_{l+1} \setminus W_l| \bigcup \bar{P}^{-1}(a)$. All the points in $\bar{P}^{-1}(a)$ have the same potential a, so φ_c is identically zero on the edges contained in $\bar{P}^{-1}(a)$. Then

$$(10) \qquad |c|_1 = \int_{|\Gamma|} |\varphi_c| = \sum_{l=0}^{\infty} \int_{|W_{l+1}\setminus W_l|} |\varphi_c| + \int_{|\bar{P}^{-1}(a)|} |\varphi_c| = \sum_{l=0}^{\infty} \int_{|W_{l+1}\setminus W_l|} |\varphi_c|,$$

so the last series converges since c is summable. Putting equalities (9) and (10) together we obtain

$$\lim_{k\to\infty} |c-c_k|_1 = \lim_{k\to\infty} \sum_{l=k}^{\infty} \int_{|W_{l+1}\setminus W_l|} |\varphi_c| = 0.$$

This finishes the proof of Lemma 17 and Proposition 14. □

5. 3-manifolds.

This section is devoted to proving the following theorem.

Theorem 18. *If G is the fundamental group of a closed 3-manifold, then $\bar{H}_n^{(1)}(G,\mathbb{R}) = 0$ for any $n \ge 1$.*

Proof. Let M be the closed 3-manifold whose fundamental group is G.

Step 1. *First we prove the theorem assuming that M is orientable, closed, and prime.* Triangulate M and put the triangulation on the universal cover $\widetilde{M}$ induced by the projection $\widetilde{M} \to M$.

If M is a 2-sphere bundle over a circle, then $G = \mathbb{Z}$, and the theorem follows since $\mathbb{Z}$ admits a $K(G,1)$ of dimension 1. So we can assume that M is not a 2-sphere bundle over a circle, and M is prime. In this case M is irreducible [7, Lemma 3.13], therefore, by the sphere theorem, $\pi_2(M)$ is trivial.

If G is finite, then $\widetilde{M}$ has a finite triangulation, and the theorem obviously follows since any summable chain in $\widetilde{M}$ has finite support. So we can also assume that G is infinite. In this case $\pi_3(M) = \pi_3(\widetilde{M}) = H_3(\widetilde{M},\mathbb{Z}) = 0$.

Since $\pi_2(M)$ and $\pi_3(M)$ are trivial, then, by Hurewicz theorem, M is a (compact) simplicial complex of type $K(G,1)$, so $\bar{H}_n^{(1)}(G,\mathbb{R}) = \bar{H}_n^{(1)}(\widetilde{M},\mathbb{R})$. Also, G has finite cohomological dimension, hence it is torsion free.

The vanishing of $\bar{H}_1^{(1)}(G,\mathbb{R})$ was shown by Allcock and Gersten in [1]. Also, $\bar{H}_3^{(1)}(G,\mathbb{R}) = 0$, because any 3-cycle on $\widetilde{M}$ must take the same (up to a sign) value on each 3-cell, and there are infinitely many 3-cells in $\widetilde{M}$, so any summable 3-cycle on $\widetilde{M}$ must be 0. It remains only to prove the theorem for $i = 2$. We need to show that any summable 2-cycle on $\widetilde{M}$ can be approximated in ℓ_1-norm by cycles of compact support.

Let Γ be the 1-skeleton of the dual cellulation on $\widetilde{M}$. Each 2-cell σ of M corresponds to an edge $f(\sigma)$ in Γ, and since M is orientable we can consistently orient the edges of Γ. This correspondence f extended by linearity gives a homomorphism $f : C_2^{(1)}(\widetilde{M},\mathbb{R}) \to C^1_{(1)}(\Gamma,\mathbb{R})$. Note that f is an isomorphism of normed spaces by definition.

Pick $z \in C_2^{(1)}(\widetilde{M},\mathbb{R})$. Then $f(z)$ is a 1-cochain in the dual cellulation of $\widetilde{M}$. For z to be a cycle is equivalent to the vanishing of $f(z)$ on 1-boundaries in the dual cellulation. These 1-boundaries are the same as 1-cycles, since $\widetilde{M}$ is simply connected. In other words, z is a summable 2-cycle on $\widetilde{M}$ if and only if $f(z)$ is a summable coboundary on Γ, i.e. f induces an isometric isomorphism

$$f : Z_2^{(1)}(\widetilde{M},\mathbb{R}) \to B^1_{sum}(\Gamma,\mathbb{R}).$$

Analogously for chains of finite support, f induces an isometric isomorphism

$$f : Z_2(\widetilde{M},\mathbb{R}) \to B^1_c(\Gamma,\mathbb{R}),$$

and we have a commutative diagram

$$\begin{array}{ccc} Z_2(\widetilde{M},\mathbb{R}) & \hookrightarrow & Z_2^{(1)}(\widetilde{M},\mathbb{R}) \\ \cong\downarrow f & & \cong\downarrow f \\ B^1_c(\Gamma,\mathbb{R}) & \hookrightarrow & B^1_{sum}(\Gamma,\mathbb{R}) \end{array}$$

where the columns are isomorphisms. Once we know that Γ is one-ended, Proposition 14 will imply $B^1_{sum}(\Gamma,\mathbb{R}) \subseteq \bar{B}^1_c(\Gamma,\mathbb{R})$, hence $Z_2^{(1)}(\widetilde{M},\mathbb{R}) \subseteq \bar{Z}_2(\widetilde{M},\mathbb{R})$, and by (5), $Z_2^{(1)}(\widetilde{M},\mathbb{R}) = \bar{Z}_2(\widetilde{M},\mathbb{R})$. The property of being one-ended is quasiisometry invariant for connected graphs, so it remains to show that the group G is one-ended.

Suppose G is not one-ended. Then the number of ends must be 0, 2, or ∞. It is not 0 in our case since the group is infinite. J. Stallings showed

Theorem 19 (Stallings [8]). *Let G be a finitely generated torsion free group. Then*

- *G has two ends if and only if $G = \mathbb{Z}$, and*
- *G has infinitely many ends if and only if G is a non-trivial free product.*

By our assumptions $G \neq \mathbb{Z}$, and in the case of infinitely many ends the splitting of G as a non-trivial free product can be realized as a connected sum $M = M_1 \sharp M_2$, where $\pi_1(M_1)$ and $\pi_1(M_2)$ are non-trivial [7, Theorem 7.1]. In particular, M is not prime, that contradicts our assumptions. This establishes *Step 1.*

Step 2. General case: M is any closed 3-manifold.

Since $\bar{H}_2^{(1)}(G,\mathbb{R})$ is quasiisometry invariant, it is not changed by passing to a subgroup of finite index. Therefore, replacing M with the double covering we can assume that M is orientable. There is a finite set of embedded spheres cutting M into prime pieces M_i, $i = 1, ..., m$. Let $\hat{M}_i$ be the closed manifold obtained by gluing 3-balls to each boundary component of $\widetilde{M}_i$. Denote $G_i := \pi_1(M_i) = \pi_1(\hat{M}_i)$. Then G decomposes as a finite graph of groups with some vertex groups Γ_i and trivial edge groups. We know from *Step 1* that $\bar{H}_2^{(1)}(G_i, \mathbb{R}) = 0$. It remains to show the following.

Lemma 20. *Let G be the fundamental group of a finite graph of groups with the trivial edge groups and such that each vertex group G_i is finitely presentable and $\bar{H}_2^{(1)}(G_i, \mathbb{R}) = 0$. Then $\bar{H}_2^{(1)}(G, \mathbb{R}) = 0$.*

Proof. Pick a $K(G_i, 1)$ complex Z_i' for each G_i. Let Z' be the cell complex obtained by connecting the complexes Z_i' according to the graph of groups decomposition. Then $\pi_1(Z') = G$ and Z' is a $K(G, 1)$ complex, so the universal cover $Z := \widetilde{Z'}$ can be used to compute $\bar{H}_2^{(1)}(G, \mathbb{R})$.

Let $\{Z_j \mid j = 1, 2, 3, ...\}$ be the set of all the lifts of complexes Z_i' to Z. If c is a summable 2-cycle in Z, it decomposes as the sum $c = \oplus_j c_j$, such that c_j is a summable 2-cycle supported on Z_j. Since each Z_j is the universal cover of some Z_i', then each c_j can be approximated by compactly supported cycles, and therefore c can be approximated by compactly supported cycles in X. This finishes Lemma 20 and Theorem 18. □ □

6. Questions.

The original question still remains open in general:

Question 1 (Gersten). *If G is a group of type $\mathcal{F}_n$, does $\bar{H}_i^{(1)}(G, \mathbb{R})$ vanish for $i \leq n$?*

Another interesting question to consider would be the following.

Question 2. *For the Thompson's group F, is there any integer $i \geq 2$ such that $\bar{H}_i^{(1)}(G, \mathbb{R}) \neq 0$? Equivalently, is there a summable i-cycle in the universal covering of a $K(F, 1)$ complex, which cannot be approximated by i-cycles of compact support?*

If the answer is "yes", then Theorem 5 will imply that the Thompson's group is not combable, in particular, not automatic.

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