HIGHER DIMENSIONAL ISOPERIMETRIC FUNCTIONS IN HYPERBOLIC GROUPS

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Abstract. We introduce the notion of an $\mathbb{R}$-combing and use it to show that hyperbolic groups satisfy linear isoperimetric inequalities for filling real cycles in *each* positive dimension. S. Gersten suggested the concept of metabolicity (over $\mathbb{Z}$ or $\mathbb{R}$) for groups which implies hyperbolicity. Metabolicity admits several equivalent definitions: by vanishing of ℓ_∞-cohomology, using combings, and others. We prove several criteria for a group to be hyperbolic, $\mathbb{R}$-metabolicity being among them. In particular, a finitely presented group G is hyperbolic iff $H^n_{(\infty)}(G,V)=0$ for any normed vector space V and any $n \geq 2$.

0. Introduction

Different versions of isoperimetric functions were discussed in various literature. A basic example of an isoperimetric function bounds the area of a disk using the length of the boundary of the disk. One can generalize this to higher dimensions (filling, say, spheres with balls), and also consider different categories: smooth manifolds, cell complexes, groups. When an edge loop in a finite cell complex can be filled with a cellular disk of area at most a linear function of the length of the loop, the fundamental group of the complex is hyperbolic. This is a very geometric concept, and it is also related to solving algorithmic problems in groups.

There is a discussion [11] on higher dimensional isoperimetric inequalities in Riemannian manifolds and in Banach spaces. Also see [4] for various kinds of isoperimetric inequalities. In [5] it is shown that for combable groups any cycle can be filled with a chain whose volume is bounded by the volume of the cycle times the diameter of the cycle. For hyperbolic groups, the linear isoperimetric inequality for filling spherical 2-cycles was shown in [2]. In the present paper we consider the homological version of isoperimetric function. It is shown, in particular, that hyperbolic groups admit linear isoperimetric functions (for $\mathbb{R}$ and $\mathbb{Q}$ coefficients) in each positive dimension.

J. Stallings characterized finitely generated free groups as groups of cohomological dimension 1 [13], or, in other words, as those groups whose second cohomology with any coefficients vanishes. This was generalized by R. Swan [14] to infinitely generated groups. S. Gersten suggested that a similar characterization may hold for hyperbolic groups in terms of ℓ_∞-cohomology. He called the groups having such a characterization *metabolic*

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(over $\mathbb{Z}$ or $\mathbb{R}$, depending on the coefficients). He also showed that metabolicity is sufficient for hyperbolicity. It was an open question if this condition was also necessary. In the present paper we prove that $\mathbb{R}$-metabolicity is necessary for hyperbolicity.

Metabolicity corresponds to vanishing of the second ℓ_∞-cohomology. Moreover, we show that the ℓ_∞-cohomology with coefficients in any vector space of any hyperbolic group vanishes in *all* dimensions ≥ 2. This is equivalent to having a linear isoperimetric inequality for filling real cycles of each positive dimension.

The following theorem is the main result of the present paper. It is a part of a larger Theorem 20 which summarizes the statements of the paper and results by Gersten [7, 10, 8] and Allcock-Gersten [1] (see the next section for definitions).

Theorem 0. *For a finitely presented group G, the following statements are equivalent.*

(a) *G is hyperbolic.*
(b) *G admits a quasigeodesic $\mathbb{R}$-combing with bounded areas.*
(c) *G is of type $\mathcal{F}_\infty$ and $H^n_{(\infty)}(G, V) = 0$ for any $n \geq 2$ and any normed real vector space V.*
(d) *G satisfies linear isoperimetric inequalities for (compactly supported) real cycles in each positive dimension.*

In particular, we answer in the affirmative the following question raised by S. Gersten in [10, p.1062] *for real chains*:

Problem. Do hyperbolic groups satisfy the linear isoperimetric inequality for n-cycles for all $n \geq 1$?

The most interesting equivalences in Theorem 0 are $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (d)$, as it was mentioned above. The author's main contributions are the use of $\mathbb{R}$-combings and implications $(a) \Rightarrow (b)$ (dandelion construction, section 3) and $(b) \Rightarrow (c)$ (which is the existence of linear isoperimetric inequalities for higher dimensions, proved in [12]). The rest is putting together known results and techniques.

The following result known before ([4], [9]) can be obtained as a corollary of the main theorem.

Corollary 22. *If M is a closed triangulated manifold which admits a metric of negative sectional curvature, then linear isoperimetric inequalities are satisfied for filling real simplicial cycles of any positive dimension on $\widetilde{M}$.*

It was asked in [1] if, given a graph Γ, there is a bounded linear retraction for the inclusion homomorphism $Z_1(\Gamma, \mathbb{R}) \subseteq C_1(\Gamma, \mathbb{R})$, where both spaces are equipped with the ℓ_1-norm. The argument in the present paper (dandelion construction) gives the affirmative answer in the case when Γ is a Cayley graph of a hyperbolic group.

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1. Definitions and known results

For a group G and $n \in \mathbb{Z}_+ \cup \{\infty\}$, $\mathcal{U}_n(G)$ will denote the set of cell complexes X which are universal covers of complexes of type $K(G,1)$ with finitely many cells in dimensions at most n. Equivalently, X is a contractible cell complex with a free cellular G-action whose restriction to the n-skeleton is cocompact. This notation is introduced only for convenience. The phrase "there exists $X \in \mathcal{U}_n(G)$" just means that G is of type $\mathcal{F}_n$.

A normed abelian group $(A, |\cdot|)$ is an abelian group A with a function $|\cdot| : A \to \mathbb{R}$ such that (1) $|a| \geq 0$ with $|a| = 0$ iff $a = 0$, for $a \in A$, (2) $|a+b| \leq |a| + |b|$ for $a, b \in A$, and (3) $|-a| = |a|$ for all $a \in A$. A normed real vector space $(V, |\cdot|)$ is a vector space V over $\mathbb{R}$ with a function $|\cdot| : V \to \mathbb{R}$ such that (1) $|v| \geq 0$ with $|v| = 0$ iff $v = 0$, for $v \in V$, (2) $|v + w| \leq |v| + |w|$ for $v, w \in V$, and (3) $|\alpha v| \leq |\alpha| \cdot |v|$ for $\alpha \in \mathbb{R}$, $v \in V$.

A chain a is a filling of a chain b if $\partial a = b$.

Definition 1. *Let X be a cell complex. The ℓ_1-norm $|\cdot|_1$ on the space of cellular chains $C_i(X, \mathbb{R})$, is defined by*

$$|\sum_\sigma \alpha_\sigma \sigma|_1 := \sum_\sigma |\alpha_\sigma|.$$

For a boundary $b \in B_i(X, \mathbb{R})$, the (real) filling norm of b is defined by

$$|b|_f := inf\{|a|_1 \mid a \in C_{i+1}(X, \mathbb{R}) \text{ and } \partial a = b\}.$$

Analogously, if $b \in B_i(X, \mathbb{Z})$, then the (integral) filling norm is

$$|b|_f := inf\{|a|_1 \mid a \in C_{i+1}(X, \mathbb{Z}) \text{ and } \partial a = b\}.$$

This definition may be confusing because an integral boundary may be viewed as a real boundary as well. In this paper we will use the real filling norm most of the time.

As shown in [7], in the case when $X \in \mathcal{U}_{i+1}(G)$,

$$|\cdot|_f \geq A|\cdot|_1 \tag{1}$$

on $B_i(X, \mathbb{R})$ for some universal constant $A > 0$. In particular, $|\cdot|_f$ is indeed a norm in this case.

Our convention is that **we always equip the space $\boldsymbol{B_i}$ with the filling norm and the spaces $\boldsymbol{C_i}$ and $\boldsymbol{Z_i}$ with the $\boldsymbol{\ell_1}$-norm**, even though Z_i and B_i may coincide as vector spaces when X is contractible.

Definition 2. *Let X be a contractible cell complex. A function $g : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is called a homological isoperimetric function for real i-cycles or an isoperimetric function, for shortness, if $|\cdot|_f \leq g(|\cdot|_1)$ on $Z_i(X, \mathbb{R}) = B_i(X, \mathbb{R})$.*

Note that formula (1) says that the linear inequality opposite to the isoperimetric inequality is satisfied in most interesting cases.

Remark. Isoperimetric function in dimension i is a quasiisometry invariant [6]. Therefore it makes sense to talk about a group having certain isoperimetric functions, namely, we say that G has a linear isoperimetric function for real i-cycles, or G satisfies a linear isoperimetric inequality for real i-cycles if, for some (and hence, for any) $X \in \mathcal{U}_{i+1}(G)$, there exists a constant $K \geq 0$ such that $|\cdot|_f \leq K|\cdot|_1$ on $Z_i(X, \mathbb{R})$.

Any edge path p connecting a vertex v to a vertex w can be viewed as an integral 1-chain with boundary $w - v$. Abusing notation, we will usually use the same letter for the edge path and the 1-chain, taking sums of paths etc. The support of a chain $c \in C_i(X, \mathbb{R})$, $supp(c)$, is the closure of the union of all i-cells σ of X with $c(\sigma) \neq 0$. (Note that, in general, the support of an edge path viewed as a chain may be a proper subset of the image of the path.)

By $i(e)$ and $t(e)$ we always denote the initial and the terminal vertices of an edge e, respectively, and we will always put the path metric d on the 1-skeleta of cell complexes, assigning length 1 to each edge.

Definition 3. *A map $f : (Y, d) \to (Y', d')$ between two metric spaces is a quasiisometric embedding if there exist constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that*

$$\frac{1}{\lambda} d(x, y) - \epsilon \leq d(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$$

for any $x, y \in Y$.

A map $f : Y \to Y'$ between two metric spaces is called a quasiisometry if there is a map $f' : Y' \to Y$ such that f and f' are quasiisometric embeddings and the maps $f' \circ f$ and $f \circ f'$ are uniformly close to identities.

A quasigeodesic path is a quasiisometric embedding of an interval of the real line.

Definition 4. *Given a cell complex X with a basepoint vertex $*$, a combing on X is an assignment of an edge path p_v connecting $*$ to v for each vertex v in X. A combing $\{p_v\}$ is called quasigeodesic if there are numbers (λ, ϵ) such that each p_v is a (λ, ϵ)-quasigeodesic path.*

We say that a group G admits a combing with bounded areas if there exist a cell complex $X \in \mathcal{U}_2(G)$, a constant $T \geq 0$, and a combing $\{q_w\}$ on X such that, for any edge e in X, $|q_{i(e)} + e - q_{t(e)}|_f \leq T$.

Definition 5. *Given a cell complex X with a basepoint vertex $*$, an $\mathbb{R}$-combing on X is an assignment of a chain (an $\mathbb{R}$-path) $q_w \in C_1(X, \mathbb{R})$ to each vertex w of X so that $\partial q_w = w - *$. An $\mathbb{R}$-combing is called quasigeodesic if there exist $S \leq 0$ and a quasigeodesic combing $\{p_w \mid w \in X^{(0)}\}$ on $X^{(1)}$ such that, for any vertex v, $supp(q_w)$ lies in the S-neighborhood of $supp(p_w)$.*

A group G admits an $\mathbb{R}$-combing with bounded areas if there exists an $\mathbb{R}$-combing such that the same condition as for combings holds, where the filling norm is taken over $\mathbb{R}$.

Definition 6. *A finitely presented group G is called hyperbolic if the following equivalent conditions are satisfied:*

(1) ***Linear isoperimetric inequality.*** *For any simply connected cell 2-complex Y with a free cocompact G-action there exists a constant $K \geq 0$ such that for any*

edge loop γ in $Y^{(1)}$ there exists a van Kampen diagram D with boundary γ and $area(D) \leq K \cdot length(\gamma)$.

(2) ***Fine triangles.*** *Given a connected graph Γ with a free cocompact G-action, there exists a constant δ such that for each geodesic triangle $[a,b],[b,c],[c,a]$ the following condition is satisfied: if $c' \in [a,b]$, $a' \in [b,c]$, $b' \in [c,a]$ is the (unique) choice of three points with*

$$d(a,b') = d(a,c'), \quad d(b,a') = d(b,c'), \quad d(c,a') = d(c,b'),$$

and $x \in [a,b]$, $y \in [a,c]$ satisfy

$$d(a,x) = d(a,y) \leq d(a,b'),$$

then $d(x,y) \leq \delta$.

See [3] for the proof of the equivalence.

Given a group G of type $\mathcal{F}_n$, a normed abelian group A, and $X \in \mathcal{U}_n(G)$, $C_i^{(1)}(X,A)$ is the vector space of the i-chains of finite ℓ_1-norm $|\cdot|_1$, where $|c|_1 := \sum_e |c(e)|$ and e runs over the i-cells of X. Since the action of G on $X^{(n)}$ is cocompact, the boundary homomorphisms $\hat{\partial}_i : C_i^{(1)}(X,A) \to C_{i-1}^{(1)}(X,A)$ are defined for $i \leq n$. The ℓ_1-homology of G is defined as the homology of the chain complex $(C_*^{(1)}(X,A), \partial)$ for $i \leq n-1$, i.e. $H_i^{(1)}(G,\mathbb{R}) := Z_i^{(1)}(X,A)/B_i^{(1)}(X,A)$, $i \leq n-1$, where $Z_i^{(1)}(X,A) := Ker\ \hat{\partial}_i$ and $B_i^{(1)}(X,A) := Im\ \hat{\partial}_{i+1}$.

$C^i_{(\infty)}(X,A)$ is the set of all cellular i-cochains on X with coefficients in the normed abelian group A which are bounded (as functions on i-cells with respect to the norm on A). The coboundary homomorphism $\delta_i : C^i_{(\infty)}(X,A) \to C^{i+1}_{(\infty)}(X,A)$ is defined for $i \leq n-1$, so $(C^*_{(\infty)}(X,A), \delta)$ is a cochain complex in dimensions $i \leq n$ and its homology $H^*_{(\infty)}(X,A)$ is defined for dimensions $i \leq n$. It is called the ℓ_∞-cohomology of G. The ℓ_∞-cohomology and the ℓ_1-homology of G are well defined since they are independent of the choice of X.

Definition 7. *A finitely presented group is called metabolic, or $\mathbb{Z}$-metabolic, if $H^2_{(\infty)}(G,A) = 0$ for any normed abelian group A. A finitely presented group is called $\mathbb{R}$-metabolic if $H^2_{(\infty)}(G,V) = 0$ for any normed real vector space V.*

This definition may seem rather abstract unless one knows the following equivalent description of metabolicity (see Theorem 12.9 in [10]).

Theorem 8 (Gersten [10])**.** *For a finitely presented group G the following conditions are equivalent.*

1. *G is metabolic.*
2. *$H^2_{(\infty)}(X, B_1(X,\mathbb{Z})) = 0$ for some $X \in \mathcal{U}_2(G)$.*
3. *G admits a combing with bounded areas.*

(4) *There exist* $X \in \mathcal{U}_2(G)$ *and a bounded additive retraction* $\rho : C_1(X,\mathbb{Z}) \to B_1(X,\mathbb{Z})$ *for the inclusion* $i : B_1(X,\mathbb{Z}) \hookrightarrow C_1(X,\mathbb{Z})$. *Here* $C_1(X,\mathbb{Z})$ *is given the* ℓ_1*-norm and* $B_1(X,\mathbb{Z})$ *the filling norm.*
(5) *There exist* $X \in \mathcal{U}_2(G)$ *and a bounded additive splitting* $s : B_0(X,\mathbb{Z}) \to C_1(X,\mathbb{Z})$ *for the boundary map* $\partial : C_1(X,\mathbb{Z}) \to B_0(X,\mathbb{Z})$.

One of the goals of the present paper is to show that "hyperbolic" implies "$\mathbb{R}$-metabolic". The implication in the other direction is known: it immediately follows from the following theorem.

Theorem 9 (Gersten [7]). *A finitely presented group* G *is hyperbolic iff* $H^2_{(\infty)}(G,\ell_\infty) = 0$.

Also, a homological characterization holds:

Theorem 10 (Allcock-Gersten [1]). *A finitely presented group* G *is hyperbolic iff* $H_1^{(1)}(G,\mathbb{R}) = 0$.

2. Some proofs

We sketch a proof of the statement analogous to Theorem 8 for the category of real vector spaces. Essentially, it is a rewriting of the argument in [10].

Theorem 11. *For a finitely presented group* G*, the following conditions are equivalent.*
(1) G *is* $\mathbb{R}$*-metabolic.*
(2) $H^2_{(\infty)}(X, B_1(X,\mathbb{R})) = 0$ *for some* $X \in \mathcal{U}_2(G)$.
(3) G *admits an* $\mathbb{R}$*-combing with bounded areas.*
(4) *There exist* $X \in \mathcal{U}_2(G)$ *and a bounded linear retraction* $\rho : C_1(X,\mathbb{R}) \to B_1(X,\mathbb{R})$ *for the inclusion* $i : B_1(X,\mathbb{R}) \hookrightarrow C_1(X,\mathbb{R})$.

Sketch of the proof. Pick $X \in \mathcal{U}_\infty(G)$. The universal cocycle u is by definition the boundary homomorphism $\partial : C_2(X,\mathbb{R}) \to B_1(X,\mathbb{R})$ restricted to the set of (oriented) 2-cells in X. It is obviously a bounded function, so u can be viewed as an element in $C^2_{(\infty)}(X, B_1(X,\mathbb{R}))$. One checks that u is indeed a cocycle.

The universal cocycle u satisfies the property that, given a normed vector space V and a cocycle $u' \in C^2_{(\infty)}(X,V)$, then u' factors through u. Therefore vanishing of $H^2_{(\infty)}(X,V)$ for any V (i.e. $\mathbb{R}$-metabolicity) follows from

$$(*) \qquad u = \delta c \text{ for some } c \in C^1_{(\infty)}(X, B_1(X,\mathbb{R})).$$

A diagram chasing argument proves that $(*)$ and (4) are equivalent. So we have **(4)** $\Leftrightarrow$ $\mathbf{(*)} \Rightarrow$ **(1)** $\Rightarrow$ **(2)** $\Rightarrow \mathbf{(*)}$, i.e. all these statements are equivalent.

(4) $\Rightarrow$ **(3)** Pick a path p_v in the 1-skeleton of X from the basepoint $*$ to v (or, more generally, a real 1-chain p_v with $\partial p_v = v - *$), and define

$$q_v := p_v - \rho(p_v).$$

If p'_v is another path from $*$ to v, then, using the fact that ρ is identity on 1-cycles,

$$(p_v - \rho(p_v)) - (p'_v - \rho(p'_v)) = (p_v - p'_v) - \rho(p_v - p'_v) = 0.$$

So the definition of q_v does not depend on the choice of path p_v. By the definition of q_v,

$$\rho(q_v) = q_v - q_v = 0.$$

Also,

$$\partial q_v = \partial p_v - \partial(\rho(p_v)) = \partial p_v,$$

so $\{q_v\}$ is an $\mathbb{R}$-combing. Let v and w be the endpoints of an edge e. Then

$$|q_v + e - q_w|_f = |\rho(q_v + e - q_w)|_f = |\rho(q_v) + \rho(e) - \rho(q_w)|_f = |\rho(e)|_f$$

is bounded independently of the choice of e, since ρ is a bounded map. Therefore $\{q_v\}$ is an $\mathbb{R}$-combing with bounded areas.

(3) $\Rightarrow$ (4) Given an $\mathbb{R}$-combing $\{q_v\}$ with bounded areas in $X \in \mathcal{U}_2(G)$, we have

$$|q_{i(e)} + e - q_{t(e)}|_f \leq T$$

for some universal constant T, where $i(e)$ and $t(e)$ are the (initial and terminal) vertices of (an oriented edge) e. Define

$$\rho(e) := q_{i(e)} + e - q_{t(e)}$$

and extend ρ by linearity to a map $\rho : C_1(X, \mathbb{R}) \to B_1(X, \mathbb{R})$.

If $z = \sum_e \alpha_e e$ is a 1-cycle, then

$$\begin{aligned}\rho(z) = \quad & \sum_e \alpha_e(q_{i(e)} + e - q_{t(e)}) = \sum_e \alpha_e e + \sum_e (q_{i(e)} - q_{t(e)}) = \\ & z + \sum_{v \in X^{(0)}} q_v\Big(\sum_{i(e)=v} \alpha_e - \sum_{t(e)=v} \alpha_e\Big) = z,\end{aligned}$$

so ρ is a retraction for the inclusion $i : B_1(X, \mathbb{R}) \to C_1(X, \mathbb{R})$.

It is bounded because, for any 1-chain $c = \sum_e \alpha_e e$,

$$|\rho(c)|_f = \left|\sum_e \alpha_e \rho(e)\right|_f \leq \sum_e |\alpha_e| \cdot |\rho(e)|_f \leq T \sum_e |\alpha_e| = T|c|_1.$$

Theorem 11 is established. □

Theorem 12. *If G is a hyperbolic group then G admits a quasigeodesic $\mathbb{R}$-combing with bounded areas.*

Theorem 12 is a significant part of the present paper, and an essential step in its proof is

3. Growing dandelions

Start with a hyperbolic group G and $X \in \mathcal{U}_\infty(G)$. Such a complex X exists by [5] and because hyperbolic groups are combable. Fix a basepoint vertex $*$ in X. The 1-skeleton $X^{(1)}$ of X has the path metric d defined by assigning length 1 to each edge in $X^{(1)}$. By $S(r)$ (respectively, $B(r)$) we denote the sphere (respectively, the ball) of radius r in $X^{(1)}$ centered at the basepoint, and by $B_v(r)$ the ball of radius r centered at v. Let $S := \bigcup_{k=0}^{\infty} S(2Dk)$, where D is a big enough universal constant, say, the fine-triangle hyperbolicity constant δ plus 5 will work. The *flower* of a vertex $v \in S(2Dk)$ is $Fl_v := B_v(D) \cap S(2Dk)$.

Fix a geodesic combing $\{p_v\}$ on X. For each $v \in S(2Dk)$ we construct a 1-chain (an "$\mathbb{R}$-path") q_v with $\partial q_v = v - *$, inductively on k. For the only point $*$ in $S(0)$ define q_* to be the zero chain (or, equivalently, the constant path $*$). Suppose $\{q_w\}$ is constructed for any vertex w in $\bigcup_{i=0}^{k-1} S(2Di)$ and let v be a vertex in $S(2Dk)$. For each $x \in Fl_v$, pick a geodesic path α_x (of length at most D) connecting x to v and let β_x be the part of the path p_x in the geodesic combing connecting $\bar{x} := supp(p_x) \cap S(2D(k-1))$ to x. The $\mathbb{R}$-path $q_{\bar{x}}$ is given by the induction hypotheses. Put

$$q_v := \frac{1}{\#Fl_v} \sum_{x \in Fl_v} (\alpha_x + \beta_x + q_{\bar{x}})$$

(see Fig. 1). Then

$$\partial q_v = \frac{1}{\#Fl_v} \sum_{x \in Fl_v} \partial(\alpha_x + \beta_x + q_{\bar{x}}) = \frac{1}{\#Fl_v} \sum_{x \in Fl_v} [(v - x) + (x - \bar{x}) + (\bar{x} - *)] = v - *,$$

i.e. q_v is an $\mathbb{R}$-path from $*$ to v. For any other vertex w in X the "one-level-lower projection" $w \mapsto \bar{w}$ is defined as above: $\bar{w}$ is the intersection point $p_w \cap S(2Di)$, where i is the maximal integer satisfying $2Di < d(w, *)$. Analogously, β_w is the part of the path p_w connecting $\bar{w}$ to w. We define $q_w := \beta_w + q_{\bar{w}}$. Again, $\partial q_w = w - *$, so $\{q_w\}$ is an $\mathbb{R}$-combing on X.

We want to prove that the constructed $\mathbb{R}$-combing is with bounded areas.

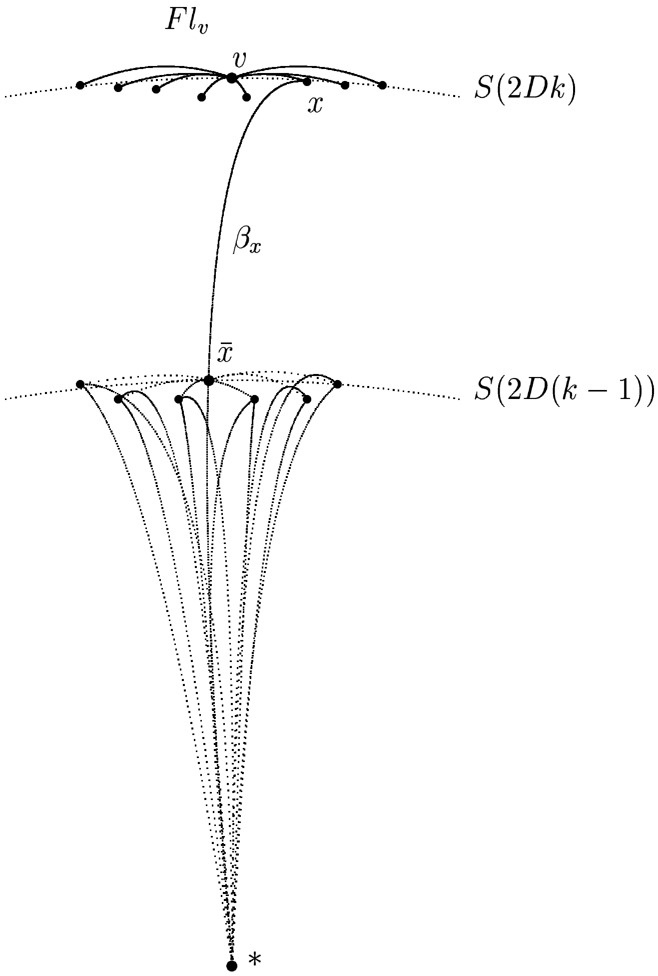
Lemma 13 (Projection property). *For a hyperbolic group and X as above the following properties hold.*

1. *If $x, y \in S(2D(k+1))$, $k \in \mathbb{Z}_+$, $d(x,y) \le 3D$, and x', y' are the intersection points of $S(2Dk)$ with some geodesics connecting $*$ to x, y, respectively, then $d(x', y') \le D$.*
2. *Let $d(x,y) \le 1$ and x', y' be points lying on some geodesic paths connecting $*$ to x, y, respectively, and suppose $d(*, x') = d(*, y')$. Then $d(x', y') \le D$.*

Proof. This lemma is an easy consequence of the fine-triangle definition of hyperbolicity. □

Proof of Theorem 12. Going from the vertex v to the basepoint along the chain q_v and applying projection property 13(1) at each step we see that the support of q_v lies in

FIGURE 1. Growing a dandelion from hight $2D(k-1)$ to hight $2Dk$.



the $4D$-neighborhood of the support of the geodesic p_v, hence the $\mathbb{R}$-combing $\{q_v\}$ is quasigeodesic.

Now we are going to show the "bounded areas" condition, i.e. that for any edge e with the initial and terminal points v and w, respectively, $|q_v + e - q_w|_f$ is bounded by a universal constant. Allcock and Gersten show in [1] that, given any graph Γ and a summable 1-cycle z, then z can be represented as a sum of coherent integral loops with real coefficients, i.e. $z = \sum_i \alpha_i z_i$, where $\alpha_i \in \mathbb{R}_+$, z_i is an edge loop, and $|z|_1 = \sum_i \alpha_i |z_i|_1$. Since hyperbolicity is equivalent to the linear isoperimetric inequality for filling (integral) loops, each z_i can be filled by an integral 2-chain c_i with $|c_i|_1 \leq K|z_i|_1$ where $K \geq 0$ is a universal constant. Thus $\sum_i \alpha_i c_i$ is a filling of z and

$$|z|_f \leq \left|\sum_i \alpha_i c_i\right|_1 \leq \sum_i \alpha_i |c_i|_1 \leq \sum_i \alpha_i K |z_i|_1 = K|z|_1.$$

In other words, in a hyperbolic group, the filling norm on $Z_1(X,\mathbb{R})$ is dominated by the ℓ_1-norm, and also

$$|q_v + e - q_w|_1 \leq |q_v - q_w|_1 + |e|_1 = |q_v - q_w|_1 + 1.$$

So it suffices only to bound $|q_v - q_w|_1$ by a universal constant.

Suppose first $v, w \in S$. We consider the following two cases.

Case 1. *For some $k \in \mathbb{Z}_+$, $v, w \in S(2Dk)$ and $d(v,w) \leq D$.*

Pick a constant $T' \geq 6D \cdot [\#B(D)]^2$. We show $|q_v - q_w|_1 \leq T'$ by induction on k.

$$|q_v - q_w|_1 = \left| \frac{1}{\#Fl_v} \sum_{x \in Fl_v} (\alpha_x + \beta_x + q_{\bar{x}}) - \frac{1}{\#Fl_w} \sum_{y \in Fl_w} (\alpha_y + \beta_y + q_{\bar{y}}) \right|_1 =$$

$$= \left| \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{x \in Fl_v} \sum_{y \in Fl_w} (\alpha_x + \beta_x + q_{\bar{x}}) - \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{y \in Fl_v} \sum_{x \in Fl_w} (\alpha_y + \beta_y + q_{\bar{y}}) \right|_1 \leq$$

$$\leq \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{x \in Fl_v} \sum_{y \in Fl_w} |\alpha_x + \beta_x + q_{\bar{x}} - \alpha_y - \beta_y - q_{\bar{y}}|_1 \leq$$

$$\leq \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{x \in Fl_v} \sum_{y \in Fl_w} [|\alpha_x + \beta_x|_1 + |\alpha_y + \beta_y|_1] + \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{x \in Fl_v} \sum_{y \in Fl_w} |q_{\bar{x}} - q_{\bar{y}}|_1 \leq$$

$$\leq 6D + \frac{1}{\#Fl_v \cdot \#Fl_w} \sum_{x \in Fl_v} \sum_{y \in Fl_w} |q_{\bar{x}} - q_{\bar{y}}|_1 \leq$$

Since $x \in Fl_v$ and $y \in Fl_w$, $d(x,y) \leq d(x,v) + d(v,w) + d(w,y) \leq 3D$, then, by projection property 13(1), $d(\bar{x}, \bar{y}) \leq D$. The induction hypotheses imply that each term $|q_{\bar{x}} - q_{\bar{y}}|_1$ in the last sum is bounded by T', and the term corresponding to $x = y = v \in Fl_v \cap Fl_w$ is zero, so, continuing the sequence of inequalities,

$$\leq 6D + \frac{1}{\#Fl_v \cdot \#Fl_w} T'(\#Fl_v \cdot \#Fl_w - 1) \leq 6D + T' - \frac{T'}{\#Fl_v \cdot \#Fl_w} \leq T'.$$

The last inequality holds because, by the choice of T',

$$6D \cdot \#Fl_v \cdot \#Fl_w \leq 6D \cdot [\#B(D)]^2 \leq T'.$$

Case 2. *For some $k \in \mathbb{Z}_+$, $v \in S(2Dk)$, $w \in S(2D(k+1))$, and v lies on a geodesic path connecting $*$ to w.*

In this case

$$|q_v - q_w|_1 = \left| q_v - \frac{1}{\#Fl_w} \sum_{y \in Fl_w} (\alpha_y + \beta_y + q_{\bar{y}}) \right|_1 =$$

$$= \frac{1}{\#Fl_w} \left| \sum_{y \in Fl_w} (q_v - \alpha_y - \beta_y - q_{\bar{y}}) \right|_1 \leq \frac{1}{\#Fl_w} \sum_{y \in Fl_w} (|q_v - q_{\bar{y}}|_1 + |\alpha_y + \beta_y|_1) \leq$$

Both v and $\bar{y}$ lie in $S(2Dk)$ and $d(w, y) \leq D$, hence by projection property $d(v, \bar{y}) \leq D$. Applying Case 1 to the vertices v, $\bar{y}$ we obtain

$$\leq \frac{1}{\#Fl_w} \sum_{y \in Fl_w} (T' + 3D) = T' + 3D.$$

General case. *$v, w \in X^{(0)}$ and $d(v, w) = 1$.*

In the general case there are again two subcases to consider.

Subcase 1. *For some $k \in \mathbb{Z}_+$, $v \in S(2D(k+1)-1)$, $w \in S(2D(k+1))$, and $d(v, w) = 1$.*

The concatenation of the path p_v and the edge connecting v to w is easily seen to be a geodesic path because its length is $2D(k+1) = d(*, w)$. Then Case 2 applies to the vertices $w \in S(2D(k+1))$ and $\bar{v} \in S(2Dk)$, so

$$|q_v - q_w|_1 \leq |\beta_v|_1 + |q_{\bar{v}} - q_w|_1 \leq 2D + (T' + 3D) = T' + 5D.$$

Subcase 2. *v and w are vertices in X such that $d(w, *) \geq d(v, *)$, $d(v, w) = 1$, and Subcase 1 does not occur.*

In this case the one-level-lower projections $\bar{v}$ and $\bar{w}$ of v and w, respectively, lie on the same level $S(2Dk)$ for some $k \in \mathbb{Z}_+$, then, by projection property 13(2), $d(\bar{v}, \bar{w}) \leq D$, and Case 1 applies to $\bar{v}$ and $\bar{w}$:

$$|q_v - q_w|_1 = |(\beta_v + q_{\bar{v}}) - (\beta_w + q_{\bar{w}})|_1 \leq |q_{\bar{v}} - q_{\bar{w}}|_1 + |\beta_v|_1 + |\beta_w|_1 \leq$$
$$\leq T' + 2D + 2D = T' + 4D.$$

So we have shown that $|q_v - q_w|_1$ is bounded by $T := T' + 5D$ whenever $d(v, w) = 1$. Theorem 12 is proved. ☐

The following result describes filling in a certain dimension $n \geq 1$. Here X denotes the universal cover of a $K(G, 1)$ complex with finite $(n+1)$-skeleton.

Theorem 14 (Gersten [8]). *The following statements are equivalent for a hyperbolic group G and integer $n \geq 2$.*

1. (1) $H_n^{(1)}(G, \mathbb{R}) = 0$.
2. (2) $H^{n+1}_{(\infty)}(G, \ell_\infty) = 0$.
3. (3) $H^{n+1}_{(\infty)}(G, \mathbb{R})$ *"vanishes strongly", i.e. there exists $K > 0$ such that for any $F \in Z^{n+1}_{(\infty)}(X, \mathbb{R})$ there exists $H \in C^n_{(\infty)}(X, \mathbb{R})$ such that $\partial H = F$ and $|H|_\infty \leq K|F|_\infty$.*
4. (4) *G satisfies the linear isoperimetric inequality for fillings of real n-cycles on X.*

We are going to show that all these properties are implied by the analogous property in dimension 1 (which, in turn, is equivalent to hyperbolicity of a group).

4. Quasiisometry invariance of combings

In this section we will prove that having combings and $\mathbb{R}$-combings with certain properties is a quasiisometry invariant. This will be needed in the next section for characterizations of hyperbolic groups, and it is interesting as a fact by itself, giving anoter geometric property of groups.

Two cell complexes will be called quasiisometric if their 1-skeleta with the path metric are quasiisometric.

Definition 15. *An ϵ_1-net in a complex Y is a subset $N \subseteq Y^{(0)}$ with the property that for any $x \in X^{(0)}$ there exists $y \in N$ such that $d(x,y) \le \epsilon_1$. A net in Y is an ϵ_1-net for some $\epsilon_1 \ge 0$. A subset $S \subseteq Y^{(0)}$ is called ϵ_2-separated net if for any pair of distinct vertices $x, y \in S$, $d(x,y) \ge \epsilon_2$. A net is separated if it is ϵ_2-separated for some $\epsilon_2 > 0$.*

Definition 16. *Given an ϵ-net N in Y, an $\mathbb{R}$-combing on (N,X) is an assignment of a 1-chain $q_v \in C_1(Y,\mathbb{R})$ to each $v \in N$ such that $\partial q_v = v - *$. Such an $\mathbb{R}$-combing on (N,Y) is with bounded areas if there exists $T \ge 0$ such that $|q_v + \gamma_{v,w} - q_w|_f \le T$ whenever $v, w \in N$, $d(v,w) \le 2\epsilon + 1$, and $\gamma_{v,w}$ is a geodesic edge path in $Y^{(1)}$ connecting v to w.*

Lemma 17. *Let N be a net in a cell 2-complex Y. Then*

1. *Y has an $\mathbb{R}$-combing with bounded areas if and only if there exists an $\mathbb{R}$-combing on (N,Y) with bounded areas.*
2. *Y has a quasigeodesic $\mathbb{R}$-combing with bounded areas if and only if there exists a quasigeodesic $\mathbb{R}$-combing on (N,Y) with bounded areas.*

Proof. **(1)** For the "only if" direction, let N be an ϵ-net and $\{q_v\}$ be an $\mathbb{R}$-combing with bounded areas on Y. Consider its restriction to N. Whenever vertices $v, w \in N$ with $d(v,w) \le 2\epsilon + 1$ are connected by a geodesic edge path $\gamma_{v,w}$ which is a consecutive concatenation of edges e_j, $j = 1, 2, ..., d(v,w) - 1$, we have

$$|q_v + \gamma_{v,w} - q_w|_f = \left| \sum_{j=0}^{d(v,w)-1} [q_{i(e)} - q_{t(e)}] + \sum_{j=0}^{d(v,w)-1} e_j \right|_f \le$$

$$\le \sum_{j=0}^{d(v,w)-1} |q_{i(e)} + e_j - q_{t(e)}|_f \le T d(v,w) \le T(2\epsilon + 1).$$

For the "if" direction, given an $\mathbb{R}$-combing $\{q_v\}$ on (N,X) with bounded areas, where N is an ϵ-net in Y, and given any vertex v in Y, pick a vertex v' in N nearest to v, and connect v' to v by a geodesic $\gamma_{v',v}$. Define $q_v := \gamma_{v',v} + q_{v'}$. Then, for any two vertices $v, w \in Y^{(0)}$ connected by an edge e, and a geodesic $\gamma_{v',w'}$ from v' to w',

$$|q_v + e - q_w|_f \le |\gamma_{v',v} + e - \gamma_{w',w} - \gamma_{v',w'}|_f + |q_{v'} + \gamma_{v',w'} - q_{w'}|_f \le |\gamma_{v',v} + e - \gamma_{w',w} - \gamma_{v',w'}|_f + T.$$

There are only finitely many (integral) cycles of form $\gamma_{v',v} + e - \gamma_{w',w} - \gamma_{v',w'}$, up to equivariance, so the last expression is bounded.
(2) Follows from the same construction. The result is still a quasiisometric combing since only short "initial segments" are added to combings. □

Lemma 18. *Let G and G' be quasiisometric groups acting freely and cocompactly on simply connected 2-dimensional cell complexes Y and Y', respectively. Then*

(1) *Y admits an $\mathbb{R}$-combing with bounded areas iff Y' does,*
(2) *Y admits a quasigeodesic $\mathbb{R}$-combing with bounded areas iff Y' does.*

Proof. **(1)** Suppose Y admits an $\mathbb{R}$-combing $\{q_v \mid v \in Y^{(0)}\}$ with bounded areas. It suffices to construct an $\mathbb{R}$-combing $\{q'_v \mid v \in Y'^{(0)}\}$ with bounded areas on Y'. The action of G and G' induces a quasiisometry $f : Y \to Y'$. The 0-skeleton $Y^{(0)}$ is a net in Y. For any $\epsilon > 0$ we can choose an ϵ-separated net $N \subseteq Y^{(0)}$. (Pick a maximal ϵ-separated subset in $Y^{(0)}$, then it is necessarily an ϵ-net.) Then $f(N)$ is an ϵ'-net in Y' for some $\epsilon' \geq 0$. Take ϵ large enough so that the restriction $f|_N$ is injective. By Lemma 17, the pair (N, Y) admits an $\mathbb{R}$-combing with bounded areas, and it suffices to show that $(f(N), Y')$ does as well.

For each pair of vertices x, y in Y' pick a geodesic path $\gamma_{x,y}$ connecting x to y so that $\gamma_{x,y} = -\gamma_{y,x}$. Do the same for the vertices in Y. The assignment $f_1(e) := \gamma_{f(i(e)),f(t(e))}$ for each edge e in Y, extended by linearity yields a bounded homomorphism $f_1 : C_1(Y, \mathbb{R}) \to C_1(Y', \mathbb{R})$, because $|\gamma_{f(i(e)),f(t(e))}|_1$ is bounded independently of e.

Analogously, one can choose a minimal filling (over $\mathbb{Z}$) of the 1-cycle $\gamma_{x,y} + \gamma_{y,z} + \gamma_{z,x}$ for each triple of vertices x, y, z in Y'. This filling exists because Y' is simply connected. So again we have a bounded homomorphism $f_2 : C_2(Y, \mathbb{R}) \to C_2(Y', \mathbb{R})$. By construction, the maps f, f_1, f_2 induce a bounded chain map $f_* : C_*(Y, \mathbb{R}) \to C_2(Y', \mathbb{R})$ in dimensions 0, 1, and 2. Define an $\mathbb{R}$-combing $\{q'_v\}$ on $(f(N), Y')$ to be the image of the $\mathbb{R}$-combing on (N, Y), i.e. $q'_v := f_*(q_{f^{-1}(v)})$ where $v \in f(N)$ and $f^{-1}(v)$ is the unique preimage of v in N.

The set $f(N)$ is an ϵ'-net. Whenever $v', w' \in f(N)$ and $d(v, w) \leq 2\epsilon + 1$, the vertices $v := f^{-1}(v')$ and $w := f^{-1}(w')$ are within a uniform distance form each other, and the norms

$$|q_v - \gamma_{v,w} - q_w|_f$$

are uniformly bounded. Then

$$\begin{aligned} |q'_{v'} - \gamma_{v',w'} - q'_{w'}|_f &\leq |q'_{v'} - f_*(\gamma_{v,w}) - q'_{w'}|_f + |f_*(\gamma_{v,w}) - \gamma_{v',w'}|_f = \\ &= |f_*(q_v) - \gamma_{v,w} - q_w)|_f + |f_*(\gamma_{v,w}) - \gamma_{v',w'}|_f \end{aligned}$$

is bounded uniformly over all such v and w, since f_* is bounded and there are at most finitely many cycles $f_*(\gamma_{v,w}) - \gamma_{v',w'}$ up to equivariance.

Thus $\{q'_v \mid v \in f(N)\}$ is an $\mathbb{R}$-combing with bounded areas on $(f(N), Y')$.
(2) Apply the same construction as in (1) and use the fact that the property of having a quasigeodesic combing is preserved via quasiisometries. □

5. Characterizations of hyperbolic groups

The following result has occurred in [12].

Theorem 19. *Let G be a finitely presented group. If G admits a quasigeodesic $\mathbb{R}$-combing with bounded areas, then there exists $X \in \mathcal{U}_\infty(G)$ such that each inclusion map $B_i(X, \mathbb{R}) \to C_i(X, \mathbb{R})$, $i \geq 1$, admits a bounded linear retraction. In particular, G has a linear isoperimetric function for real cycles in each positive dimension.*

Roughly, the idea of the proof was the following. If G admits an $\mathbb{R}$-combing with bounded areas, theorems 11 and 10 imply that G is hyperbolic. In particular, G is combable in the sense of [5] (i.e. G admits a quasigeodesic combing with the fellow-traveler property), an it is shown there that in this case there exists $X \in \mathcal{U}_\infty(G)$. As we saw in section 4, having a quasigeodesic $\mathbb{R}$-combing with bounded areas is a quasiisometry invariant property, so X admits an $\mathbb{R}$-combing $\{q_w\}$ with bounded areas as well. (Quasiisometry invariance of $\mathbb{R}$-combings was not proved in [12], so we presented it in section 4 of this paper, for completeness.) We "cone off" cells to the basepoint using this quasigeodesic $\mathbb{R}$-combing. Each i-cell σ and the "cone" over $\partial\sigma$ form an i-cycle. We fill this cycle with volume bounded by the volume of the cycle times a universal constant. In order to do this we slice the cycle starting from the basepoint, fill each slice with a linear bound, and then "fill the gaps" between the fillings.

The theorem below summarizes the results of the present paper and those by Gersten [7, 10, 8] and Allcock-Gersten [1]. Recall that B_i is always assumed to be the normed vector space with the filling norm, and C_i is the one with the ℓ_1-norm.

Theorem 20. *For a finitely presented group G, the following statements are equivalent.*

(1) *G is hyperbolic.*
(2) *G admits an $\mathbb{R}$-combing with bounded areas.*
(3) *G admits a quasigeodesic $\mathbb{R}$-combing with bounded areas.*
(4) *There exist $X \in \mathcal{U}_2(G)$ and a bounded linear retraction $\rho : C_1(X, \mathbb{R}) \to B_1(X, \mathbb{R})$ for the inclusion map $i : B_1(X, \mathbb{R}) \hookrightarrow C_1(X, \mathbb{R})$.*
(5) *There exists $X \in \mathcal{U}_\infty(G)$ such that for any $k \geq 1$ there exists a bounded linear retraction $\rho : C_k(X, \mathbb{R}) \to B_k(X, \mathbb{R})$ for the inclusion map $i : B_k(X, \mathbb{R}) \hookrightarrow C_k(X, \mathbb{R})$.*
(6) *$H^2_{(\infty)}(X, B_1(X, \mathbb{R})) = 0$ for some $X \in \mathcal{U}_2(G)$.*
(7) *$H^2_{(\infty)}(G, V) = 0$ for any normed real vector space V.*
(8) *G is of type $\mathcal{F}_\infty$ and $H^n_{(\infty)}(G, V) = 0$ for any $n \geq 2$ and any normed real vector space V.*
(9) *$H^{(1)}_1(G, \mathbb{R}) = 0$.*
(10) *G is of type $\mathcal{F}_\infty$ and $H^{(1)}_k(G, \mathbb{R}) = 0$ for any $k \geq 1$.*
(11) *G satisfies the linear isoperimetric inequality for (compactly supported) real 1-cycles.*
(12) *G satisfies linear isoperimetric inequalities for (compactly supported) real cycles in each positive dimension.*

Proof of Theorem 20.

Implication **(1)** $\Rightarrow$ **(3)** is the conclusion of Theorem 12.

Implications **(3)** $\Rightarrow$ **(2)**, **(5)** $\Rightarrow$ **(4)**, **(8)** $\Rightarrow$ **(7)**, **(10)** $\Rightarrow$ **(9)**, and **(12)** $\Rightarrow$ **(11)** are obvious.

The equivalence **(7)** $\Leftrightarrow$ **(6)** $\Leftrightarrow$ **(2)** $\Leftrightarrow$ **(4)** is the content of Theorem 11.

(3) $\Rightarrow$ **(5)** $\Rightarrow$ **(12)** is Theorem 19.

(9) $\Leftrightarrow$ **(1)** is Theorem 10.

(7) $\Rightarrow$ **(1)** follows from Theorem 9.

(11) $\Leftrightarrow$ **(1).** Theorem 5.1(2) in [7] implies that (11) is equivalent to vanishing of $H^2_{(\infty)}(G, \ell_\infty)$, then apply Theorem 9.

(7) $\Rightarrow$ **(10)** $\Leftrightarrow$ **(12).** The above implications show that each of the statements (10), (7), and (12) implies (1), that is, hyperbolicity of G. Hence the implications (7) $\Rightarrow$ (10) $\Leftrightarrow$ (12) follow from Theorem 14.

(4) $\Rightarrow$ **(7)** is shown by the following diagram-chasing argument. Let f be any bounded V-valued 2-cocycle in X. Then f extends by linearity to a bounded linear functional $f : C_2(X, \mathbb{R}) \to V$. By the definition of a cocycle, f vanishes on $B_2(X, \mathbb{R}) = Z_2(X, \mathbb{R}) \subseteq C_2(X, \mathbb{R})$, hence f factors as a composition of $\partial : C_2(X, \mathbb{R}) \to B_1(X, \mathbb{R})$ and some $f' : B_1(X, \mathbb{R}) \to V$. By the assumptions in (4), the restriction of $\rho : C_1(X, \mathbb{R}) \to B_1(X, \mathbb{R})$ to $B_1(X, \mathbb{R})$ is identity, hence $f' \circ \rho : C_1(X, \mathbb{R}) \to V$ is an extension of f'. This extension is bounded as a composition of two bounded maps, so the restriction of $f' \circ \rho$ to the set of 1-cells is bounded, i.e. this restriction is an element of $C^1_{(\infty)}(X, V)$. Also, because ρ is identity on $B_1(X, \mathbb{R})$, $f = f' \circ \partial = f' \circ \rho \circ \partial = \delta(f' \circ \rho)$, i.e. f represents zero in $H^2_{(\infty)}(X, V)$.

(5) $\Rightarrow$ **(8)** is proved by the same argument as (4) $\Rightarrow$ (7), considering dimension $n = k + 1$ instead of 2.

Now it is a tedious exercise to see that these implications actually prove Theorem 20. □

To clear the geometry behind these formal arguments, note once again that the linear isoperimetric inequalities follow directly from the existence of a quasigeodesic $\mathbb{R}$-combing with bounded areas, as we saw at the end of the last section. Essentially, it is just "coning off" real cycles using the $\mathbb{R}$-combing, so that the "volume" of the cone is bounded by a multiple of the "volume" of the cycle.

An amazing corollary of theorems 20 and 9 is that the vanishing of $H^2_{(\infty)}(G, \ell_\infty)$ implies the vanishing of $H^n_{(\infty)}(G, V)$ for any $n \geq 2$ and any normed real vector space V. The implication (2) $\Rightarrow$ (3) in Theorem 20 is also quite interesting.

It is worth mentioning the following two immediate corollaries of Theorem 20.

Corollary 21. *A finitely presented group G is hyperbolic if and only if G is $\mathbb{R}$-metabolic.*

Corollary 22. *If M is a closed triangulated manifold which admits a metric of negative sectional curvature, then linear isoperimetric inequalities are satisfied for filling real simplicial cycles of any positive dimension on $\widetilde{M}$.*

The last corollary was proved in [4]. Also see [9] for a proof and a discussion on this matter.

Note that everywhere in this paper $\mathbb{R}$ could be replaced by $\mathbb{Q}$. Though $\mathbb{Z}$-coefficients are different. The question whether any hyperbolic group is $\mathbb{Z}$-metabolic still remains open.

There is an intuitive difference between metabolicity and hyperbolicity. If c is a filling of a 1-cycle a, one may think of c as a "cone" over a. Hyperbolicity then means that the area of the "cone" with *some* vertex is linearly bounded by the length of the cycle a. The cone vertices may differ for different cycles. Metabolicity says that all cycles can be "coned off" with respect to a cone vertex, *one* for all cycles. It is rather surprising that over $\mathbb{R}$ these two properties imply each other.

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