METRIC CONFORMAL STRUCTURES AND HYPERBOLIC DIMENSION

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Abstract. For any hyperbolic complex X and $a \in X$ we construct a visual metric $\check{d} = \check{d}_a$ on ∂X that makes the $\mathrm{Isom}(X)$-action on ∂X bi-Lipschitz, Möbius, symmetric and conformal.

We define a stereographic projection of $\check{d}_a$ and show that it is a metric conformally equivalent to $\check{d}_a$.

We also introduce a notion of hyperbolic dimension for hyperbolic spaces with group actions. Problems related to hyperbolic dimension are discussed.

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1. Introduction.

This paper deals with the notions of conformal structure and conformal dimension of metric spaces, most notably ideal boundaries of hyperbolic spaces. This subject has rich history, and it seems to be an impossible task to track all its origins and contributions made by various people. We will give some references, without any hope for a complete list. The author is very much obliged to those who participated in the discussion (see the list at the end of this introduction).

In [24], Pansu attached a conformal structure to the ideal boundary of a manifold of negative curvature. Bourdon [4] described a conformal structure on the ideal boundary of a CAT(-1) space. These spaces are defined by comparison to the standard hyperbolic space $\mathbb{H}^n$ (see [5] for definitions), and therefore allow for the use of hyperbolic trigonometry. Considering (quasi)conformal structures at infinity for groups started with the works of Margulis [19] and Floyd [12]. This area has been developed quite extensively since. In particular, a metric quasiconformal structure was known to exist for hyperbolic groups; see Ghys-de la Harpe [14].

We will work in the category of *hyperbolic complexes* by which we mean simplicial complexes whose 1-skeleton is a hyperbolic graph of uniformly bounded valence. For example, a Cayley graph of a hyperbolic group can be viewed as such. The CAT(-1) property is quite restrictive, for example any CAT(-1) space must be contractible. In contrast, hyperbolic complexes can be arbitrarily bad locally; they are much more general and often occur in practice, for example Cayley graphs of non-free hyperbolic groups do not admit CAT(-1) metrics. One can also find many higher-dimensional simplicial examples, for instance various coverings of triangulated manifolds. Another interesting "example" is the connected sum $M\#K$, where M is a closed hyperbolic 3-manifold and K is a *hypothetical* counterexample to the Poincaré conjecture.

The goal of the present paper is to show that the ideal boundaries of hyperbolic complexes admit metric conformal structures with all the necessary sharp properties, just as in the CAT(-1) case.

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The existence of conformal structures is of interest in particular because of the relation to the Cannon's conjecture ([6],[8],[9],[7]) that states that a hyperbolic group Γ with $\partial\Gamma$ homeomorphic to the 2-sphere admits a proper cocompact action on $\mathbb{H}^3$. This conjecture originated from (but is not immediately implied by) the Thurston's geometrization conjecture. In [6] Cannon proved a combinatorial Riemann mapping theorem, and then using the Sullivan-Tukia results [26], [28], Cannon and Swenson [9] showed that for a hyperbolic Γ the existence of a proper cocompact Γ-action on $\mathbb{H}^3$ is equivalent to a certain combinatorial conformality property for balls in $\partial\Gamma$. Bonk and Kleiner ([2], [1], [3]), used methods of geometric analysis to address the question. They prove the Cannon's conjecture under any of the following three assumptions: when (a) $\partial\mathcal{G}$ is Ahlfors 2-regular, when (b) $\partial\Gamma$ is Ahlfors Q-regular and Q-Loewner in the sense of Heinonen and Koskela [17] for some $Q \geq 2$, or when (c) the metric on $\partial\Gamma$ quasisymmetrically equivalent to an Ahlfors Q-regular metric with Ahlfors regular conformal dimension Q. They work with quasi-Möbius maps and quasisymmetries; in the present paper we show that hyperbolic groups and complexes admit "dequasified" versions of these notions (Theorem 14 and Definition 11).

Pick a basepoint $a \in X$. The following two ways to define a metric on the boundary ∂X has been known.

(a) For the function $\mu(x) := e^{-\epsilon d(x,a)}$, Gromov [15, 7.2.K] and Coornaert-Delzant-Papadopoulos [10] define the μ-length of a path in ∂X by integrating μ along the path, and let the distance between x and y in ∂X be the infimum of all μ-lengths of paths from x to y in X.

(b) In the second approach, Ghys-de la Harpe [13, §3] use the *Gromov product*

$$(x|y)_a := \frac{1}{2}\big(d(a,x) + d(a,y) - d(x,y)\big), \qquad a, x, y \in X. \tag{1}$$

(See also a simple argument left to the reader in [15, 1.8.B].) This product usually has no continuous extension to the compactification $\bar{X} := X \cup \partial X$, as simple examples show (see [5], Definition 3.15 and Example 3.16). The construction of the metric is two-step: for $x, y \in \partial X$ and $\epsilon > 0$ small enough, one first defines

$$\rho_a(x,y) := \rho_{a,\epsilon}(x,y) := e^{-\epsilon (x|y)_a} \tag{2}$$

and then lets

$$d_a(x,y) := d_{a,\epsilon}(x,y) := \inf \sum_{i=1}^{n} \rho_{a,\epsilon}(x_{i-1}, x_i), \tag{3}$$

where the infimum is taken over all chains $x = x_0, x_1, \ldots, x_n = y$. Usually the function ρ_a is not a metric on ∂X (see Lemma 7 below), so the second step is necessary.

Recall that a map $f : (Z, d) \to (Z', d')$ is *quasiconformal* if there exists $K \in [1, \infty)$ such that for each $x \in Z$,

$$\limsup_{r \to 0} \frac{\sup\{d'(fx, fy) \mid y \in Z \text{ and } d(x, y) \le r\}}{\inf\{d'(fx, fz) \mid z \in Z \text{ and } d(x, z) \ge r\}} \le K \tag{4}$$

(cf. [13, Ch. 7, §4]). Each isometry g of a hyperbolic complex X induces a homeomorphism of ∂X which is Lipschitz and quasiconformal with respect to $d_{a,\epsilon}$ defined above ([13, Ch. 7, §4]).

In [22] a metric $\hat{d}$ was constructed for any hyperbolic group. A slightly modified version of $\hat{d}$, defined on X, was used in [21] to construct a metric on the symmetric join of $\bar{X}$ with sharp properties. In particular, $\hat{d}$ is $\mathrm{Isom}(X)$-invariant. It was also shown in [21] that if one defines the Gromov product using $\hat{d}$ instead of the word metric, i.e.

$$\langle x|y\rangle_a := \frac{1}{2}\big(\hat{d}(a, x) + \hat{d}(a, y) - \hat{d}(x, y)\big), \tag{5}$$

then $\langle\cdot|\cdot\rangle_a$ extends continuously and therefore canonically to a function $\bar{X}^2 \to [0, \infty]$.

Suppose (X, d) is a CAT(-1) space. If $(\cdot|\cdot)_a$ is defined as in (1), Bourdon showed in [4] that for any $\epsilon \in (0, 1]$, the formula $d_a(x, y) := e^{-\epsilon(x|y)_a}$ gives a metric on ∂X, that is, in the CAT(-1) case the second step is unnecessary. [5, p.435] says "however one cannot construct visual metrics on the boundary of arbitrary hyperbolic spaces in such a direct manner". One result of the present paper is that such a direct construction is nevertheless possible for hyperbolic groups and complexes *if* one uses $\hat{d}$: we define

$$\check{d}(x, y) := \check{d}_a(x, y) := \check{d}_{a,\epsilon}(x, y) := e^{-\epsilon\langle x|y\rangle_a}, \tag{6}$$

where $\langle x|y\rangle_a$ is as in (5). We use the convention $e^{-\infty} = 0$. (The results in [5] are correct; the above quote from [5] is only to emphasize that the results of the present paper are new.)

Theorem 5. *Let X be a hyperbolic complex. There exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ and every $a \in X$, the function $\check{d}$ defined above is a metric on ∂X.*

The metric $\check{d}$ on ∂X is therefore obtained from $\hat{d}$ in one step, just as in the CAT(-1) case. Note that $\check{d}$ is actually well-defined as a function on $\bar{X}^2$, though it is a metric only on ∂X^2. We show that $\check{d}_a$ is Hölder equivalent and quasiconformally equivalent to d_a (Theorem 6).

Given a hyperbolic complex X, $\hat{d}$ was used in [21] to define a continuous horofunction $\beta^\times$ on the larger space $\circledast X$ (the symmetric join of X). In section 6 we consider its restriction $\hat{\beta} : \bar{X} \times X^2 \to \mathbb{R}$ and give a simpler proof that it is Lipschitz in each variable (Proposition 10).

We consider strong notions of *conformality* and *symmetry* (Definition 11). An advantage of the definition (6) is that it provides a family of metrics that is $\mathrm{Isom}(X)$-invariant, Lipschitz, symmetric, and conformal (Theorem 13). The continuity of $\hat{\beta}$ is used in proving the conformality of this family.

Define the *cross-ratio* on ∂X via $\check{d}$, i.e.

$$[\![x, x'|y, y']\!] := \frac{\check{d}_a(x, y)\, \check{d}_a(x', y')}{\check{d}_a(x, y')\, \check{d}_a(x', y)}. \tag{7}$$

This expression obviously makes sense for pairwise distinct points x, x', y, y' in ∂X. It can also be defined in larger domains $\partial^\diamond X$ or $\bar{X}^\diamond$, and it is independent of the choice of a (see (9) and section 5).

Theorem 14 shows that if one puts the metric $\check{d}_a$ on the ideal boundary of a hyperbolic complex X, then the homomorphism of ∂X induced by $g \in \mathrm{Isom}(X)$ is Möbius, bi-Lipschitz, symmetric, and conformal.

In the classical case, the boundary of $\mathbb{H}^n$ is the round sphere $\mathbb{S}^{n-1}$, and for any $b \in \mathbb{S}^{n-1}$ the stereographic projection maps $\mathbb{S}^{n-1} \setminus \{b\}$ conformally onto the plane with the standard Euclidean metric. For $a \in X$ and $b \in \bar{X}$ we define the *stereographic projection* of $\check{d}_a$ with respect to b by

$$\check{d}_{a|b}(x, y) := \frac{\check{d}_a(x, y)}{\check{d}_a(x, b)\, \check{d}_a(y, b)}, \qquad x, y \in \partial X \setminus \{b\}. \tag{8}$$

This formula is the same as in elementary Euclidean geometry for the classical stereographic projection, where the Euclidean metric is replaced with $\check{d}$ (see section 9). We show that it works for any hyperbolic complex X and indeed defines a metric:

Theorem 16. *For any hyperbolic complex X, the function $\check{d}_{a|b}$ is a metric on $\partial X \setminus \{b\}$ conformally equivalent to $\check{d}_a$. The metrics $\check{d}_a$ and $\check{d}_{a|b}$ induce the same (usual) topology on $\partial X \setminus \{b\}$.*

$\check{d}_{a|b}$ generalizes the metric defined by Hersonsky-Paulin who used a different formula in the CAT(-1) case [18, p. 383]. It also strengthens the quasiconformal metric in [14].

In section 10 we define a notion of hyperbolic dimension for groups and spaces. This takes it origins in and is related to the notion of conformal dimension; the reader is advised to use the following references as a guide: Margulis [19], Pansu [24], Gromov [16, 15], Bonk-Kleiner [2], [1], [3].

Theorem 25 is a rigidity result for hyperbolic complexes: under the assumption of equivariance, the notions of conformal, symmetric and Möbius maps on the ideal boundary coincide.

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2. The metric $\check{d}_a$ on ∂X.

The ideal boundary $\partial\mathbb{H}^n$ of the standard hyperbolic space $\mathbb{H}^n$ is the round sphere $\mathbb{S}^2$. There are two natural metrics on $\mathbb{S}^n$ with respect to a basepoint $a \in \mathbb{H}^n$: the *angle metric* at a which is the same as the path metric in $\mathbb{S}^n$, or the *chordal metric*, the one induced from $\mathbb{R}^{n+1}$ and expressed as twice the sine of the half-angle. A particular choice is not important since the two metrics are conformally equivalent. In this section we prove Theorem 5 which gives an analog of the chordal metric for any hyperbolic complex X.

2.1. Definition of $\hat{d}$.

For completeness we remind the definition of the metric $\hat{d}$ on a hyperbolic complex X. See [20] for the properties of f and $\bar{f}$ defined below, and [22, sections 3 and 5] and [21, 6.1] for more details on the metric $\hat{d}$.

Let $\mathcal{G}$ be the 1-skeleton of X. We endow $\mathcal{G}$ with the word metric d, i.e. the path metric obtained by assigning each edge length 1. Let δ be a positive integer such that all the geodesic triangles in $\mathcal{G}$ are δ-fine.

The *ball* $B(x, R)$ is the set of all vertices at distance at most R from the vertex x. The *sphere* $S(x, R)$ is the set of all vertices at distance R from the vertex x. Pick a geodesic bicombing p in $\mathcal{G}$; that is a choice of a geodesic edge path $p[a, b]$ for each pair of vertices a, b in $\mathcal{G}$. By $p[a, b](t)$ we denote the point on the geodesic path $p[a, b]$ at distance t from a. $C_i(\mathcal{G}, \mathbb{Q})$ is the space of all i-chains in $\mathcal{G}$ with coefficients in $\mathbb{Q}$. Endow $C_0(\mathcal{G}, \mathbb{Q})$ with the ℓ^1-norm $|\cdot|_1$ with respect to the standard basis $\mathcal{G}^{(0)}$.

For $v, w \in \mathcal{G}^{(0)}$, the *flower* at w with respect to v is defined to be

$$Fl(v,w) := S(v, d(v,w)) \cap B(w,\delta) \subseteq \mathcal{G}^{(0)}.$$

Let $\mathrm{Geod}(a,b)$ be the finite set of all geodesic paths in $\mathcal{G}$ from a to b. View each geodesic as a 1-chain and define

$$p'[a,b] := \frac{1}{\#\mathrm{Geod}(a,b)} \sum_{s \in \mathrm{Geod}(a,b)} s \quad \in C_1(\mathcal{G}, \mathbb{Q}),$$

$$p'[a,b](t) := \frac{1}{\#\mathrm{Geod}(a,b)} \sum_{s \in \mathrm{Geod}(a,b)} s(t) \quad \in C_0(\mathcal{G}, \mathbb{Q}).$$

Define $pr_a : \mathcal{G}^{(0)} \to C_0(\mathcal{G}, \mathbb{Q})$ by:

- $pr_a(a) := a$;
- if $b \neq a$, $pr_a(b) := p'[a,b](t)$, where t is the largest integral multiple of 10δ which is strictly less than $d(a,b)$.

This extends by linearity to a $\mathbb{Q}$-linear map $pr_a : C_0(\mathcal{G}, \mathbb{Q}) \to C_0(\mathcal{G}, \mathbb{Q})$.

Now for each pair $a, b \in \mathcal{G}^{(0)}$, define a 0-chain $f(a,b)$ in $\mathcal{G}$ inductively on the distance $d(a,b)$ as follows:

- if $d(a,b) \le 10\delta$, $f(a,b) := b$;
- if $d(a,b) > 10\delta$ and $d(a,b)$ is not an integral multiple of 10δ, let $f(a,b) := f(a, pr_a(b))$;
- if $d(a,b) > 10\delta$ and $d(a,b)$ is an integral multiple of 10δ, let

$$f(a,b) := \frac{1}{\#Fl(a,b)} \sum_{x \in Fl(a,b)} f(a, pr_a(x)),$$

  where $f(a, pr_a(x))$ is defined by linearity in the second variable.

In what follows we will interchange a and b in the notation.

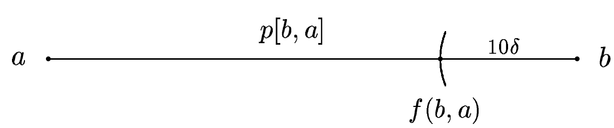

FIGURE 1. Convex combination $f(b,a)$.

For each $a \in \mathcal{G}^{(0)}$, a 0-chain $star(a)$ is defined by

$$star(a) := \frac{1}{\#B(a, 7\delta)} \sum_{x \in B(a,7\delta)} x.$$

This extends to a linear operator $star : C_0(\mathcal{G}, \mathbb{Q}) \to C_0(\mathcal{G}, \mathbb{Q})$. Define the 0-chain $\bar{f}(b, a)$ by $\bar{f}(b, a) := star\big(f(b, a)\big)$.

For each pair of vertices $a, b \in \mathcal{G}^{(0)}$, a rational number $r(a, b) \geq 0$ is defined inductively on $d(a, b)$ as follows.

- $r(a, a) := 0$.
- If $0 < d(a, b) \leq 10\delta$, let $r(a, b) := 1$.
- If $d(a, b) > 10\delta$, let $r(a, b) := r\big(a, \bar{f}(b, a)\big)+1$, where $r\big(a, \bar{f}(b, a)\big)$ is defined by linearity in the second variable.

The function r is well defined by [20, Proposition 7(2)].

For all $a, b \in \mathcal{G}^{(0)}$, let

$$s(a, b) := \frac{1}{2}\big[r(a, b) + r(b, a)\big]$$

and

$$\hat{d}(a, b) := \begin{cases} s(a, b) + C & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases}$$

where C is a sufficiently large constant depending only on X. $\hat{d}$ is an $\mathrm{Isom}(X)$-invariant metric on $\mathcal{G}^{(0)} = X^{(0)}$, and it extends to X by linearity over simplices. It is shown in [22] and [21] that $\hat{d}$ is an $\mathrm{Isom}(X)$-invariant metric on X quasiisometric to the word metric.

2.2. **The double difference $\langle \cdot, \cdot | \cdot, \cdot \rangle$.** This notion was first considered by Otal [23] for negatively curved manifolds, under the name "symplectic cross-ratio". Since in this paper we consider two metrics $\hat{d}$ and $\check{d}$, and one is obtained by exponentiating the other, it is important to clearly distinguish between sums and products. We will therefore consistently call differences differences and ratios ratios in what follows.

Let $(a, a', b, b') \in \bar{X}^4$. A *$\partial X$-triple in* (a, a', b, b') is a set of three distinct letters taken from a, a', b, b' in which each letter represents a point in $\partial X \subseteq \bar{X}$. A ∂X-triple is *trivial* if the three letters represent the same point in ∂X. Denote

$$\begin{aligned} (9)\ & \bar{X}^\diamond := \\ & \big\{(a, a', b, b') \in \bar{X}^4 \;\big|\; \text{each } \partial X\text{-triple in } (a, a', b, b') \text{ is non-trivial}\big\}, \\ & \partial^\diamond X := \\ & \big\{(a, a', b, b') \in (\partial X)^4 \;\big|\; \text{each } \partial X\text{-triple in } (a, a', b, b') \text{ is non-trivial}\big\}. \end{aligned}$$

We have $X^4 \subseteq \bar{X}^\diamond \subseteq \bar{X}^4$, with the topology on $\bar{X}^\diamond$ induced by the last inclusion; similarly for $\partial^\diamond X \subseteq (\partial X)^4$.

$\bar{\mathbb{R}} := [-\infty, \infty]$ is the two-point compactification of $\mathbb{R}$. Let $\hat{d}$ be the metric on X defined above. The *double difference in* X is the function $\langle \cdot, \cdot | \cdot, \cdot \rangle : X^4 \to \mathbb{R}$ defined by

$$\langle a, a' | b, b' \rangle := \frac{1}{2}\big(\hat{d}(a,b) - \hat{d}(a',b) - \hat{d}(a,b') + \hat{d}(a',b')\big). \tag{10}$$

One might call it the *$\hat{d}$-double difference* to emphasize the metric used in its definition. In addition we define the double difference $(\cdot, \cdot | \cdot, \cdot) : (X^{(0)})^4 \to \mathbb{R}$ with respect to the word metric d on $X^{(0)}$:

$$(a, a' | b, b') := \frac{1}{2}\big(d(a,b) - d(a',b) - d(a,b') + d(a',b')\big). \tag{11}$$

Extend $(\cdot, \cdot | \cdot, \cdot)$ to all of X^4 by linearity over simplices. One can further extend $(\cdot, \cdot | \cdot, \cdot)$ to $\bar{X}^\diamond$ by taking limits along *some* sequences of points in $X^{(0)}$; such an extension usually depends on the choice of sequences and is neither unique nor continuous.

The following was shown in [21].

Theorem 1 ([21, 6.7]). *If X is a hyperbolic complex, the double difference $\langle \cdot, \cdot | \cdot, \cdot \rangle : X^4 \to \mathbb{R}$ with respect to $\hat{d}$ defined in (10) extends to a continuous* $\mathrm{Isom}(X)$*-invariant function $\langle \cdot, \cdot | \cdot, \cdot \rangle : \bar{X}^\diamond \to \bar{\mathbb{R}}$ with the following properties.*

(a) $\langle a, a' | b, b' \rangle = \langle b, b' | a, a' \rangle$.
(b) $\langle a, a' | b, b' \rangle = - \langle a', a | b, b' \rangle = - \langle a, a' | b', b \rangle$.
(c) $\langle a, a | b, b' \rangle = 0$, $\langle a, a' | b, b \rangle = 0$.
(d) $\langle a, a' | b, b' \rangle + \langle a', a'' | b, b' \rangle = \langle a, a'' | b, b' \rangle$, *where by convention we allow $\pm\infty \mp \infty = r$ and $\pm\infty + r = \pm\infty$ for any $r \in \mathbb{R}$, and $\pm\infty \pm \infty = \pm\infty$.*
(e) $\langle a, b | c, x \rangle + \langle b, c | a, x \rangle + \langle c, a | b, x \rangle = 0$ *with the same convention.*
(f) $\langle a, a' | b, b' \rangle = \infty$ *if and only if $a = b' \in \partial X$ or $a' = b \in \partial X$.*
(g) $\langle a, a' | b, b' \rangle = -\infty$ *if and only if $a = b \in \partial X$ or $a' = b' \in \partial X$.*
(h) *For every hyperbolic complex X, the functions $\langle \cdot, \cdot | \cdot, \cdot \rangle$ and $(\cdot, \cdot | \cdot, \cdot)$ on $\bar{X}^\diamond$ are ${}^{\times +}$equivalent, i.e. there exist $A \in [1, \infty)$ and $B \in [0, \infty)$ depending only on X such that for all $(a, a', b, b') \in \bar{X}^\diamond$,*

$$\frac{1}{A}\,(a, a' | b, b') - B \le \langle a, a' | b, b' \rangle \le A\,(a, a' | b, b') + B.$$

Note also that by the triangle inequality, $|\, \langle a, a' | b, b' \rangle \,| \le \hat{d}(a, a')$.

Proposition 2 ([21, 6.8]). *For each hyperbolic complex X there exist constants $T \in [0, \infty)$ and $\lambda \in [0, 1)$ such that for all $(u, a, b, c) \in \bar{X}^\diamond$, if*

$\langle u, a|b, c\rangle \geq T$ *or* $\langle u, b|a, c\rangle \geq T$, *then*

$$|\langle u, c|a, b\rangle| \leq \lambda^{\langle u,a|b,c\rangle} \leq 1 \quad \textit{and} \quad |\langle u, c|a, b\rangle| \leq \lambda^{\langle u,b|a,c\rangle} \leq 1.$$

2.3. **The product $\langle\cdot|\cdot\rangle_a$.** Let d be the word metric on $X^{(0)}$ and define the product $(\cdot|\cdot)_a$ as in (1). One can extend $(\cdot|\cdot)_a$ to the case when some of x, y lie in ∂X by taking limits along some sequences of points in $X^{(0)}$. Such an extension in general is neither unique nor continuous.

For $a \in X$ we define another function $\langle\cdot|\cdot\rangle_a : X^2 \to [0, \infty)$ as in (5) and let

$$\bar{X}^{\triangleright} := \{(x, y, a) \in \bar{X}^3 \mid a \in \partial X \;\; \to \;\; (x \neq a \text{ and } y \neq a)\}.$$

We have $X^3 \subseteq \bar{X}^{\triangleright} \subseteq \bar{X}^3$. Since $\langle x|y\rangle_a = \langle x, a|a, y\rangle$, the following is a corollary of Theorem 1.

Theorem 3 ([21, 5.7])**.** *If X is a hyperbolic complex, the Gromov product $\langle x|y\rangle_a$ with respect to $\hat{d}$ given by (5) extends to a continuous function $\langle\cdot|\cdot\rangle_\cdot : \bar{X}^{\triangleright} \to [0, \infty]$ such that $\langle x|y\rangle_a = \infty$ iff $a \in \partial X$ or $x = y \in \partial X$.*

For a fixed $a \in X$ the above theorem implies that the product extends to a well-defined continuous function $\langle\cdot|\cdot\rangle_a : \bar{X}^2 \to [0, \infty]$.

Proposition 4. *For every hyperbolic complex X, the products $(\cdot|\cdot)_\cdot$ and $\langle\cdot|\cdot\rangle_\cdot$ defined in (1) and (5) are ${}^{\times+}$equivalent as functions on $\bar{X}^{\triangleright}$, i.e. there are constants $A \in [1, \infty)$ and $B \in [0, \infty)$ such that*

$$\frac{1}{A}(x|y)_a - B \leq \langle x|y\rangle_a \leq A\,(x|y)_a + B$$

for all $(x, y, a) \in \bar{X}^{\triangleright}$.

Proof. This follows from Theorem 1(h) since $\langle x|y\rangle_a = \langle x, a|a, y\rangle$ and $(x|y)_a = (x, a|a, y)$. ☐

3. The metric $\check{d}_a$.

For $\epsilon > 0$ and $a \in X$, define as in (6) the function

$$\check{d} = \check{d}_a : \bar{X} \times \bar{X} \to [0, 1] \qquad \text{by} \qquad \check{d}_a(x, y) := e^{-\epsilon\langle x|y\rangle_a}.$$

Note that $\check{d}$ is defined on pairs of points in $\bar{X}$, not just in ∂X, but $\check{d}$ is not a metric on $\bar{X}$ since $\check{d}(x, x) > 0$ for $x \in X$.

Theorem 5. *Let X be a hyperbolic complex. There exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ and every $a \in X$, the function $\check{d}$ defined above is a metric on ∂X.*

Proof. Our assumption is that $a \in X$. Let $\lambda \in [0,1)$ and $T \in [0,\infty)$ be the constants from Proposition 2. Increasing λ and T if needed, we can assume that $\lambda \in [e^{-1},1)$. Choose $\epsilon_0 > 0$ small enough so that

$$\epsilon_0 T \le \ln 2, \qquad \epsilon_0 \lambda^T \le \ln 2 \qquad \text{and} \qquad \epsilon_0 \le -\ln\lambda. \tag{12}$$

Pick any $\epsilon \in (0,\epsilon_0]$, then by the above choice of λ and ϵ_0 we have

$$2e^{-\epsilon T} \ge 1, \qquad 2e^{-\epsilon\lambda^T} \ge 1 \qquad \text{and} \qquad e^{-\epsilon} \ge \lambda \ge e^{-1}. \tag{13}$$

First we will show the triangle inequality

$$e^{-\epsilon\langle x|y\rangle_a} + e^{-\epsilon\langle y|z\rangle_a} \ge e^{-\epsilon\langle x|z\rangle_a} \tag{14}$$

for arbitrary $x,y,z \in \bar{X}$ (not just in ∂X). Since by Theorem 3, $\langle\cdot|\cdot\rangle_a$ is continuous in $\bar{X}^2$, and X is dense in $\bar{X}$, it suffices to prove the inequality when $x,y,z \in X$. In this case all the terms involved are finite, and by the definitions (5) and (10) of $\langle\cdot|\cdot\rangle_{\cdot}$ and $\langle\cdot,\cdot|\cdot,\cdot\rangle$ in X, (14) is equivalent to each of the following:

$$e^{-\epsilon(\langle x|y\rangle_a - \langle x|z\rangle_a)} + e^{-\epsilon(\langle y|z\rangle_a - \langle x|z\rangle_a)} \ge 1,$$

$$e^{-\epsilon\langle a,x|y,z\rangle} + e^{-\epsilon\langle a,z|y,x\rangle} \ge 1. \tag{15}$$

Denote $s := \langle a,x|y,z\rangle$ and $t := \langle a,z|y,x\rangle$. Note that $e^{-\epsilon s}$ is decreasing and $e^{-\epsilon\lambda^s}$ is increasing in s; this observation will be used in the following four cases.

Case 1. $s \le T$ *and* $t \le T$.
By (13) we have $e^{-\epsilon s} + e^{-\epsilon t} \ge e^{-\epsilon T} + e^{-\epsilon T} = 2e^{-\epsilon T} \ge 1$, which proves (15).

Case 2. $s \ge T$ *and* $t \ge T$.
By Proposition 2, $s \ge T$ implies $t \le |t| \le \lambda^s$, and $t \ge T$ implies $s \le |s| \le \lambda^t$, therefore by (13),

$$e^{-\epsilon s} + e^{-\epsilon t} \ge e^{-\epsilon\lambda^t} + e^{-\epsilon\lambda^s} \ge e^{-\epsilon\lambda^T} + e^{-\epsilon\lambda^T} = 2e^{-\epsilon\lambda^T} \ge 1.$$

(Alternatively, assume that $T \ge 1$ and deduce from the inequalities that Case 2 is impossible.)

Case 3. $s \ge T$ *and* $t \le T$.
The assumptions imply that $s \ge 0$. Note that $f(v) := v + e^{-v} \ge 1$ for all $v \ge 0$ because $f(0) = 1$ and f is increasing. By Proposition 2, $s \ge T$ implies $t \le |t| \le \lambda^s$, therefore by (13),

$$\begin{aligned} e^{-\epsilon s} + e^{-\epsilon t} &\ge e^{-\epsilon s} + e^{-\epsilon\lambda^s} = (e^{-\epsilon})^s + (e^{-\epsilon})^{\lambda^s} \\ &\ge \lambda^s + (e^{-1})^{\lambda^s} = \lambda^s + e^{-\lambda^s} \ge 1. \end{aligned}$$

Case 4. $s \le T$ *and* $t \ge T$.
This is the same as Case 3 with s and t interchanged.

This shows the triangle inequality for $\check{d}$ in $\bar{X}$ (not just in ∂X). Our assumption is that $a \in X$, so for $x, y \in \partial X$, Theorem 3 implies that $\check{d}(x,y) = 0 \Leftrightarrow x = y$, hence $\check{d}$ is a metric on ∂X. $\square$

4. Comparing to the case of word metrics.

In this section we compare the metric $\check{d}$ with the one coming from the word metric. The following theorem can be deduced from the general argument as in [4] (one would only need to modify the definition of quasiconformality), since $\hat{d}$ and the word metric are quasiisometric (as shown in [22] and [21]). Below we also provide a short direct proof that is based on properties of $\hat{d}$.

Theorem 6. *There exists $\epsilon_0 > 0$ such that for every $\epsilon, \epsilon' \in (0, \epsilon_0]$ and $a, a' \in X^{(0)}$, the following hold.*

(a) *$\check{d}_{a,\epsilon}$ in (6) and $d_{a',\epsilon'}$ in (3) are Hölder equivalent. In particular, $\check{d}$ induces the usual topology on ∂X and $(\partial X, \check{d})$ is of finite Hausdorff dimension.*

(b) *$\check{d}_{a,\epsilon}$ is quasiconformally equivalent to $d_{a',\epsilon'}$, i.e. the identity maps $(\partial X, \check{d}_{a,\epsilon}) \leftrightarrows (\partial X, \check{d}_{a',\epsilon'})$ are quasiconformal. Moreover, there exists $K = K(X, \epsilon, a, \epsilon', a') < \infty$ such that for any $x \in \partial X$,*

$$\sup_{r\in(0,\infty)} \frac{\sup\{\check{d}_{a,\epsilon}(x,y) \mid y \in \partial X \text{ and } d_{a',\epsilon'}(x,y) \le r\}}{\inf\{\check{d}_{a,\epsilon}(x,z) \mid z \in \partial X \text{ and } d_{a',\epsilon'}(x,z) \ge r\}} \le K \tag{16}$$

and

$$\sup_{r\in(0,\infty)} \frac{\sup\{d_{a',\epsilon'}(x,y) \mid y \in \partial X \text{ and } \check{d}_{a,\epsilon}(x,y) \le r\}}{\inf\{d_{a',\epsilon'}(x,z) \mid z \in \partial X \text{ and } \check{d}_{a,\epsilon}(x,z) \ge r\}} \le K. \tag{17}$$

Proof. **(a)** It is shown in [13, Ch. 7, §3] that the metric $d_{a',\epsilon'}$ in (3) and the function $\rho_{a',\epsilon'}$ in (2) are Lipschitz equivalent with a constant $C = C(X, a', \epsilon') > 0$, and that $(\partial X, d_{a',\epsilon'})$ has finite Hausdorff dimension. Changing the basepoint a to a' gives Lipschitz equivalent metrics, so we will assume $a = a'$. Hölder equivalence of $\check{d}_{a,\epsilon}$ in (6) and $\rho_{a,\epsilon'}$ follows from Proposition 4. This implies that $d_{a,\epsilon'}$ and $\check{d}_{a,\epsilon}$ are Hölder equivalent. Thus $\check{d}$ defines the same topology on ∂X as the metric in (3), and $(\partial X, \check{d})$ has finite Hausdorff dimension.

(b) Choose $\epsilon_0 > 0$ so that for all $\epsilon, \epsilon' \in (0, \epsilon_0]$ and $a, a' \in X^{(0)}$, $\check{d}_{a,\epsilon}$ and $d_{a',\epsilon'}$ are metrics on ∂X. To show (16), pick any $r \in (0, \infty)$ and $x, y, z \in \partial X$ satisfying $d_{a',\epsilon'}(x,y) \le r$ and $d_{a',\epsilon'}(x,z) \ge r$. As described in (a) above, $d_{a',\epsilon'}$ and $\rho_{a',\epsilon'}$ are C-Lipschitz equivalent, hence

$$\rho_{a',\epsilon'}(x,y) \le Cr, \qquad \rho_{a',\epsilon'}(x,z) \ge r/C,$$

Since $\langle x|z\rangle_a - \langle x|y\rangle_a = \langle a,x|z,y\rangle$ and $(x|z)_a - (x|y)_a = (a,x|z,y)$, by Theorem 1(h),

$$\langle x|z\rangle_a - \langle x|y\rangle_a \le A\big((x|z)_a - (x|y)_a \big) + B \le \\ A\big((x|z)_{a'} - (x|y)_{a'} \big) + 2A\, d(a,a') + B,$$

where A and B depend only on X. Hence

$$\epsilon\,\langle x|z\rangle_a - \epsilon\,\langle x|y\rangle_a \le \frac{\epsilon A}{\epsilon'}\big(\epsilon'\,(x|z)_a - \epsilon'\,(x|y)_a\big) + 2\epsilon A\, d(a,a') + \epsilon B,$$

$$\frac{\check{d}_{a,\epsilon}(x,y)}{\check{d}_{a,\epsilon}(x,z)} \le \left(\frac{\rho_{a',\epsilon'}(x,y)}{\rho_{a',\epsilon'}(x,z)}\right)^{\epsilon A/\epsilon'} \cdot e^{2\epsilon A\, d(a,a')+\epsilon B} \\ \le C^{2\epsilon A/\epsilon'} e^{2\epsilon A\, d(a,a')+\epsilon B}.$$

Setting K to be the right hand side, this implies (16). (17) is proved similarly, with $C = 1$. ☐

From now on we will always assume that $\epsilon > 0$ is chosen so that it satisfies (13).

Recall that for $\check{d}_a$ to be *visual* means that for each $b \in X$ there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$\frac{1}{C} e^{-\varepsilon (x|y)_b} \le \check{d}_a(x,y) \le C e^{-\varepsilon (x|y)_b}. \tag{18}$$

for all $x, y \in \partial X$, $a \in X$. When one takes $b = a$ and $\langle\cdot|\cdot\rangle_a$ instead of $(\cdot|\cdot)_a$, then $\check{d}_a$ becomes "the most visual possible", that is (18) holds with $C = 1$, $\varepsilon = \epsilon$ and equalities on both sides, as follows from the definition (6). Now the triangle inequality and Proposition 4 imply (18), i.e. $\check{d}$ is visual with respect to $(\cdot|\cdot)_\cdot$ as well.

The following lemma says that if one starts with the word metric on X, a metric on ∂X cannot be defined in one step.

Lemma 7. *Suppose $\mathcal{G}$ is a hyperbolic graph, d is the word metric on $\mathcal{G}$, and $\partial\mathcal{G}$ is not totally disconnected. Then ρ defined by (2) is not a metric on $\partial\mathcal{G}$ inducing the usual topology.*

The fundamental groups of closed hyperbolic n-manifolds for $n \ge 2$, and more generally, their free products, satisfy the assumptions of the lemma, giving examples when ρ is not a metric.

Proof. If d is the word metric, the Gromov product (1) takes only values of type $k/2$, where $k \in \mathbb{Z}$. Assume that ρ is a metric inducing the topology of $\partial\mathcal{G}$. If $\partial\mathcal{G}$ is not totally disconnected, then there exist distinct points x and y in a connected component C of $\partial\mathcal{G}$. The image of C under the continuous map $z \mapsto \rho(x,z)$ must contain $\rho(x,x) = 0$ and

$\rho(x,y)$, and therefore the whole interval $[0,\rho(x,y)]$. This contradicts the fact that ρ can take only values of type $e^{-\epsilon k/2}$, $k \in \mathbb{Z}$. □

5. The cross-ratio in $\bar{X}$.

Let the *cross-ratio* on $\bar{X}$ be the function

$$(19) \qquad [\![\cdot,\cdot|\cdot,\cdot]\!] : \bar{X}^{\diamond} \to [0,\infty], \qquad [\![x,x'|y,y']\!] := e^{\epsilon\langle x,x'|y,y'\rangle},$$

with the convention $e^{-\infty}=0$ and $e^{\infty}=\infty$. Here we take the same ϵ as in the definition of $\check{d}_a$ in section 3. We will see (Proposition 9) that this definition extends (7) to the larger domain $\bar{X}^{\diamond}$. It is also clear from (19) that the cross-ratio does not depend on the choice of basepoint a.

The following is "$e^{\text{Theorem 1}}$".

Theorem 8. *The cross-ratio $[\![\cdot,\cdot|\cdot,\cdot]\!] : \bar{X}^{\diamond} \to [0,\infty]$ in (19) is a well-defined continuous* Isom(X)*-invariant function satisfying the following properties.*

(a) $[\![a,a'|b,b']\!] = [\![b,b'|a,a']\!]$.
(b) $[\![a,a'|b,b']\!] = 1/[\![a',a|b,b']\!] = 1/[\![a,a'|b',b]\!]$.
(c) $[\![a,a|b,b']\!] = 1$, $[\![a,a'|b,b]\!] = 1$.
(d) $[\![a,a'|b,b']\!] \cdot [\![a',a''|b,b']\!] = [\![a,a''|b,b']\!]$, *where by convention we allow* $\infty\cdot 0 = r$ *and* $\infty \cdot r = \infty$ *for any* $r\in[0,\infty]$, *and* $\infty\cdot\infty=\infty$.
(e) $[\![a,b|c,x]\!] \cdot [\![b,c|a,x]\!] \cdot [\![c,a|b,x]\!] = 1$ *with the same convention.*
(f) $[\![a,a'|b,b']\!] = \infty$ *if and only if* $a=b'\in\partial X$ *or* $a'=b\in\partial X$;
(g) $[\![a,a'|b,b']\!] = 0$ *if and only if* $a=b\in\partial X$ *or* $a'=b'\in\partial X$.

Proposition 9. *Suppose* $a\in X$ *and* $(x,x',y,y')\in\bar{X}^{\diamond}$. *Then*

$$[\![x,x'|y,y']\!] = e^{\langle x,x'|y,y'\rangle} = \frac{\check{d}_a(x,y)\,\check{d}_a(x',y')}{\check{d}_a(x,y')\,\check{d}_a(x',y)},$$

i.e. the two definitions of cross-ratio $[\![\cdot,\cdot|\cdot,\cdot]\!]$ *in (7) and (19) agree.*

Proof. Both sides of the equality are continuous wherever defined, and X^4 is dense in $\bar{X}^{\diamond}$, hence it suffices to show the equality only for $(x,x',y,y')\in X^4$. In this case by direct calculation we have

$$\langle x,x'|y,y'\rangle = \langle x,y\rangle_a - \langle x,y'\rangle_a - \langle x',y\rangle_a + \langle x',y'\rangle_a.$$

This and the definition of $\check{d}_a$ imply the desired equality. □

If $a',b,b'\in\bar{X}$ do not all coincide and $a\in X$, we have

$$(20) \qquad [\![a,a'|b,b']\!] = e^{-\epsilon\langle a'|b'\rangle+\epsilon\langle a'|b\rangle} = \frac{\check{d}_a(a',b')}{\check{d}_a(a',b)}.$$

Similar formulas are obtained by permuting variables.

6. Horofunctions.

Define $\hat{\beta}_u(x,y) := \hat{d}(u,x) - \hat{d}(u,y)$ for $(u,x,y) \in X^3$ and

$$\hat{\beta}_u(x,y) := \lim \hat{\beta}_v(x,y) \quad \text{as} \quad v \to u \quad \text{along X}$$

for $(u,x,y) \in \partial X \times X^2$. This limit indeed exists in $\mathbb{R}$, because by the continuity and properties of the double difference in $\bar{X}^\diamond$ (Theorem 1), for an arbitrary $v_0 \in X$,

$$\begin{aligned} \lim \hat{\beta}_v(x,y) &= \lim \big(\hat{d}(v_0,x) - \hat{d}(v_0,y) + \langle v_0, v|x,y\rangle \big) \\ &= \hat{d}(v_0,x) - \hat{d}(v_0,y) + \lim \langle v_0, v|x,y\rangle \\ &= \hat{d}(v_0,x) - \hat{d}(v_0,y) + \langle v_0, u|x,y\rangle \in \mathbb{R}, \end{aligned} \tag{21}$$

where all the limits are taken as $v \to u$ along $\bar{X}$, not just along X.

The function $\hat{\beta} : \bar{X} \times X^2 \to \mathbb{R}$ defined above is called the *horofunction* in $\bar{X}$. The continuity of the double difference and (21) imply that $\hat{\beta}$ is continuous in the three variables. This horofunction $\hat{\beta}$ is the restriction of the continuous function $\beta^\times$ that was defined in [21, 8.1] on a larger domain $\bar{X} \times (*\bar{X})^2$.

Proposition 10. *Put metrics $\hat{d}$ on X and $\check{d} = \check{d}_a$ on ∂X with respect to a fixed basepoint $a \in X$. The function $\hat{\beta} : \partial X \times X^2 \to \mathbb{R}$ is Lipschitz in each variable. Moreover, this is true in the domain $\bar{X} \times X^2$ (in the obvious sense, even though $\check{d}$ is not a metric in $\bar{X}$).*

Proof. Let $T \in [0,\infty)$ and $\lambda \in [0,1)$ be as in Proposition 2 and recall that by our choice of ϵ in (12) and (13), $\lambda \le e^{-\epsilon}$. Fix $x, y \in X$ and let

$$C := \max\{\lambda^{-\hat{d}(a,x) - \hat{d}(a,y)},\ \hat{d}(x,y)\lambda^{-T-\hat{d}(a,x)-\hat{d}(a,y)}\}.$$

Pick arbitrary $u, u' \in \partial X$. If $\langle u, x|y, u'\rangle \ge T$ then by Proposition 2,

$$\begin{aligned} |\,\langle u, u'|x,y\rangle\,| &\le \lambda^{\langle u,x|y,u'\rangle} = \lambda^{\langle u,a|a,u'\rangle + \langle a,x|a,u'\rangle + \langle u,x|a,y\rangle} \\ &\le \lambda^{\langle u|u'\rangle_a - \hat{d}(a,x) - \hat{d}(a,y)} \le C\lambda^{\langle u|u'\rangle_a} \le Ce^{-\epsilon\langle u|u'\rangle_a} \\ &= C\check{d}(u,u'). \end{aligned}$$

If $\langle u, x|y, u'\rangle \le T$ then

$$\begin{aligned} |\,\langle u, u'|x,y\rangle\,| &\le \hat{d}(x,y) \le \hat{d}(x,y)\lambda^{\langle u,x|y,u'\rangle - T} \\ &\le \hat{d}(x,y)\lambda^{\langle u|u'\rangle_a - \hat{d}(a,x) - \hat{d}(a,y) - T} \le C\lambda^{\langle u|u'\rangle_a} \\ &\le Ce^{-\epsilon\langle u|u'\rangle_a} = C\check{d}(u,u'). \end{aligned}$$

Hence

$$|\hat{\beta}_u(x,y) - \hat{\beta}_{u'}(x,y)| = |\,\langle u, u'|x,y\rangle\,| \le C\check{d}(u,u')$$

for all $u, u', a, x, y \in X$. By continuity this extends to the case when $u, u' \in \bar{X}$. This means that $\hat{\beta}_u(x, y)$ is Lipschitz in u.

By the triangle inequality,

$$|\hat{\beta}_u(x, y) - \hat{\beta}_u(x', y)| = |\hat{d}(u, x) - \hat{d}(u, x')| \leq \hat{d}(x, x')$$

holds for $u, x, x', y \in X$, and by continuity extends to the case when $u \in \bar{X}$. This means that $\hat{\beta}_u(x, y)$ is Lipschitz in x. The argument for y is similar. □

7. Conformality and symmetry.

We will use the following strong definitions of conformality and symmetry.

Definition 11. *A map $f : (Z, d) \to (Z', d')$ between two metric spaces is called metric conformal, or just conformal, at a point $x \in X$ if the limit*

$$\lim \frac{d'\big(f(x), f(y)\big)}{d(x, y)} \quad as \quad y \to x \quad along \;\; Z \setminus \{x\}$$

exists in $(0, \infty)$. The above limit, denoted $|f'(x)|$, will be called the metric derivative of f at x. The map f is conformal in Z if it is conformal at every $x \in Z$.

A map $f : (Z, d) \to (Z', d')$ is symmetric at $x \in Z$ if

$$\lim \frac{d'\big(f(x), f(y)\big)\; d(x, z)}{d'\big(f(x), f(z)\big)\; d(x, y)} \quad as \quad y, z \to x \quad along \;\; Z \setminus \{x\}$$

exists and equals 1. The map f is symmetric in Z, or is a symmetry in Z, if it is symmetric at every $x \in Z$.

Remark 1. The above notion of conformality was considered by Bourdon in [4] (and applied to boundaries of CAT(-1)-spaces). The above notion of symmetry is our "dequasified" version of the notion of quasisymmetry introduced by Tukia and Väisälä ([30]). Note that being conformal or symmetric are local properties.

Remark 2. The above definition of conformality can be strengthened by requiring that $|f'(x)|$ be a continuous function of x. The class of such homeomorphisms f would become a metric analog of the class C^1 for functions on manifolds. Theorem 14(d) below says that the homeomorphism of the ideal boundary of a hyperbolic complex X induced by an isometry of X is in this class.

We also remark on abstract nonsense. It is a part of the definition of symmetry that the ratio $\frac{d'\left(f(x),f(y)\right)\; d(x,z)}{d'\left(f(x),f(z)\right)\; d(x,y)}$, and therefore each of the four distances, is a number in $(0, \infty)$ for all y, z in some neighborhood of x. When x is isolated in Z, then, if one traces the definition of lim formally, any number is a limit. In this case "lim exists in $(0, \infty)$" should be interpreted as "there is a number in $(0, \infty)$ which is a limit", and "lim exists and equals 1" as "1 is a limit".

The following is immediate from definitions.

Lemma 12. *Conformal maps are symmetries. Compositions of conformal maps are conformal. Compositions of symmetries are symmetries. Inverses of conformal maps are conformal. Inverses of symmetries are symmetries.*

Two metrics d and d' on a topological space Z are called *conformally equivalent* if the identity map $(Z, d) \to (Z, d')$ (and hence its inverse) is conformal.

8. Properties of $\check{d}_a$.

This section shows that the metric $\check{d}$ on the boundary of a hyperbolic complex satisfies the same properties as the metric on the boundary of a CAT(-1) space (item (d) in the two theorems below is similar to [4, Corollaire 2.6.3]).

Theorem 13. *Let X be a hyperbolic complex. For a fixed ϵ, the family of functions $\{\check{d}_a \mid a \in X\}$ on $\bar{X}^2$ defined in (6) is*

(a) $\mathrm{Isom}(X)$*-invariant in the sense that $\check{d}_{ga}(gx, gy) = \check{d}_a(x, y)$ for all $g \in \mathrm{Isom}(X)$ and $x, y \in \bar{X}$,*
(b) *Lipschitz, moreover, $\check{d}_a(x, y) \le e^{\hat{d}(a,b)} \check{d}_b(x, y)$ for all $x, y \in \bar{X}$ and $a, b \in X$,*
(c) *symmetric, i.e. for all $a, b \in X$, the identity map $(\partial X, \check{d}_a) \to (\partial X, \check{d}_b)$ is symmetric, and*
(d) *conformal, i.e. for all $a, b \in X$, $\check{d}_a$ and $\check{d}_b$ are conformally equivalent as functions in $\bar{X}^2$. Moreover, for each $x \in \bar{X}$, the limit*

$$\lim \frac{\check{d}_b(x, y)}{\check{d}_a(x, y)} \quad \text{as} \quad y \to x \quad \text{along} \quad \bar{X} \setminus \{x\}$$

equals $e^{\epsilon \hat{\beta}_x(a,b)} \in (0, \infty)$.

Proof. **(a)** The double difference $\langle a, a' | b, b' \rangle$ is $\mathrm{Isom}(X)$-invariant by Theorem 1, therefore so is the product $\langle x|y \rangle_a = \langle x, a | a, y \rangle$. This implies the statement.

(b) By the continuity of $\langle\cdot|\cdot\rangle_{\cdot}$ (Theorem 3) it suffices to show the inequality for $x, y \in X$. In this case by the triangle inequality for $\hat{d}$,

$$\langle x|y\rangle_a = \langle x|y\rangle_b + \frac{1}{2}\big(\hat{d}(a,x) - \hat{d}(b,x) + \hat{d}(a,y) - \hat{d}(b,y)\big) \le \langle x|y\rangle_b + \hat{d}(a,b),$$

which implies $\check{d}_a(x,y) \le e^{\hat{d}(a,b)}\check{d}_b(x,y)$.
(c) follows from (d) below.
(d) Fix any $a, b \in X$, then for all $x, y \in X$ we have

$$\langle x|y\rangle_a - \langle x|y\rangle_b = \frac{1}{2}\big(\hat{d}(x,a) - \hat{d}(x,b) + \hat{d}(y,a) - \hat{d}(y,b)\big)$$
$$= \frac{1}{2}\big(\hat{\beta}_x(a,b) + \hat{\beta}_y(a,b)\big)$$

and by the triangle inequality,

$$|\hat{\beta}_y(a,b)| \le \hat{d}(a,b) < \infty.$$

By the continuity of $\hat{\beta}$ and $\langle\cdot|\cdot\rangle_{\cdot}$ all the above hold for $x, y \in \bar{X}$. Now fix any $x \in \bar{X}$, then as $y \to x$ along $\bar{X} \setminus \{x\}$,

$$\lim\big(\langle x|y\rangle_b - \langle x|y\rangle_a\big) = \lim \frac{1}{2}\big(\hat{\beta}_x(a,b) + \hat{\beta}_y(a,b)\big) = \hat{\beta}_x(a,b),$$

and therefore by the definition of $\check{d}$,

$$\lim \frac{\check{d}_b(x,y)}{\check{d}_a(x,y)} = \lim e^{\epsilon(\langle x|y\rangle_b - \langle x|y\rangle_a)} = e^{\epsilon\hat{\beta}_x(a,b)} \in (0,\infty). \tag{22}$$

□

Theorem 14. *Let X be a hyperbolic complex. Put the metric $\check{d} = \check{d}_a$ defined in (6) on ∂X. Then for any $g \in \mathrm{Isom}(X)$, the homeomorphism induced by g on $(\partial X, \check{d})$ is*

(a) *Möbius, i.e. $[\![gx, gx'|gy, gy']\!] = [\![x, x'|y, y']\!]$ for all $(x, x', y, y') \in \partial X^{\diamond}$ (see (9)), where the cross-ratio is defined in terms of $\check{d}$,*
(b) *bi-Lipschitz with constant $e^{\hat{d}(a,g^{-1}a)}$,*
(c) *symmetric, and*
(d) *conformal, moreover, for all $x \in \bar{X}$, the derivative*

$$|g'(x)| := \lim \frac{\check{d}(gx,gy)}{\check{d}(x,y)} \quad \text{as } y \to x \text{ along } \bar{X} \setminus \{x\}$$

equals $e^{\epsilon\hat{\beta}_x(a,g^{-1}a)} \in (0,\infty)$.
(e) *Furthermore, for all $x, y \in \partial X$,*

$$|g'(x)|\,|g'(y)|\,\check{d}(x,y)^2 = \check{d}(gx,gy)^2.$$

Proof. **(a)** The cross-ratio $[\![\cdot,\cdot|\cdot,\cdot]\!]$ is $\mathrm{Isom}(X)$-invariant by Theorem 8.
(b) follows from Theorem 13(ab):

$$\check{d}_a(gx,gy)=\check{d}_{g^{-1}a}(x,y)\le e^{\hat{d}(a,g^{-1}a)}\check{d}_a(x,y).$$

(c) follows from (d) below.
(d) Take $g\in\mathrm{Isom}(X)$, $a\in X$, $x\in\bar{X}$. By Theorem 13(ad), as $y\to x$ along $\bar{X}\setminus\{x\}$,

$$\lim\frac{\check{d}_a(gx,gy)}{\check{d}_a(x,y)}=\lim\frac{\check{d}_{g^{-1}a}(x,y)}{\check{d}_a(x,y)}=e^{\epsilon\hat{\beta}_x(a,g^{-1}a)}\in(0,\infty).$$

(e) If $x=y$ the statement is obvious, so we assume otherwise.
"$x'\to x$" will mean "$x'\to x$ along $\partial X\setminus\{x\}$", and similarly for $y'\to y$.
By direct calculation,

$$\frac{e^{-2\epsilon\langle x|y\rangle_a}}{e^{-\epsilon\langle x|x'\rangle_a}\,e^{-\epsilon\langle y|y'\rangle_a}}=e^{\epsilon\langle a,x|x',y\rangle-\epsilon\langle a,y|x,y'\rangle}$$

for all $x,x',y,y'\in X$, and therefore by continuity this holds for all $x,x',y,y'\in\partial X$ with $x\neq y$.

$$\begin{aligned}
&\frac{|g'(x)|\,|g'(y)|\,\check{d}(x,y)^2}{\check{d}(gx,gy)^2}\\
&=\lim_{x'\to x}\frac{d(g(x),g(x'))}{d(x,x')}\lim_{y'\to y}\frac{d(g(y),g(y'))}{d(y,y')}\frac{e^{-2\epsilon\langle x|y\rangle_a}}{e^{-2\epsilon'\langle g(x')|g(y')\rangle_a}}\\
&=\lim_{x'\to x,\ y'\to y}\left(\frac{e^{-\epsilon'\langle g(x)|g(x')\rangle_a}}{e^{-\epsilon\langle x|x'\rangle_a}}\frac{e^{-\epsilon'\langle g(y)|g(y')\rangle_a}}{e^{-\epsilon\langle y|y'\rangle_a}}\frac{e^{-2\epsilon\langle x|y\rangle_a}}{e^{-2\epsilon'\langle g(x')|g(y')\rangle_a}}\right)\\
&=\lim_{x'\to x,\ y'\to y}\frac{e^{\epsilon\langle a,x|x',y\rangle-\epsilon\langle a,y|x,y'\rangle}}{e^{\epsilon\langle g(a),x|x',y\rangle-\epsilon\langle g(a),y|x,y'\rangle}}\\
&=\lim_{x'\to x,\ y'\to y}e^{\epsilon\langle a,g(a)|x',y\rangle-\epsilon\langle a,g(a)|x,y'\rangle}=1.
\end{aligned}$$

□

Definition 15. *Given a hyperbolic space X with an isometric Γ-action, a conformal structure for (X,Γ) is an invariant conformal family of metrics on ∂X, i.e. a family $\{\check{d}_a\mid a\in X\}$ such that*

1. $\check{d}_{ga}(gx,gy)=\check{d}_a(x,y)$ *for* $g\in\Gamma$, $a\in X$, $x,y\in\partial X$, *and*
2. *for all* $a,b\in X$, $\check{d}_a$ *and* $\check{d}_b$ *are conformally equivalent in the sense of section 7.*

The above theorems say that for any hyperbolic complex X, $(X,\mathrm{Isom}(X))$ admits a conformal structure.

9. Stereographic projection: the metric $\check{d}_{a|b}$ on $\partial X \setminus \{b\}$.

9.1. **The standard stereographic projection.** Consider the sphere $\mathbb{S}^n$ of radius $1/2$ centered at $(0,\ldots,0,1/2) \in \mathbb{R}^{n+1}$. Let d_E be the Euclidean metric on $\mathbb{R}^{n+1}$. The *chordal metric* on $\mathbb{S}^n$ is the restriction of d_E to $\mathbb{S}^n$, also denoted d_E. Let d'_E be the metric induced on $\mathbb{S}^n \setminus \{b\}$ from the hyperplane $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ via the inverse stereographic projection $\mathbb{R}^n \to \mathbb{S}^n$ with respect to $b := (0,\ldots,0,1) \in \mathbb{S}^n$.

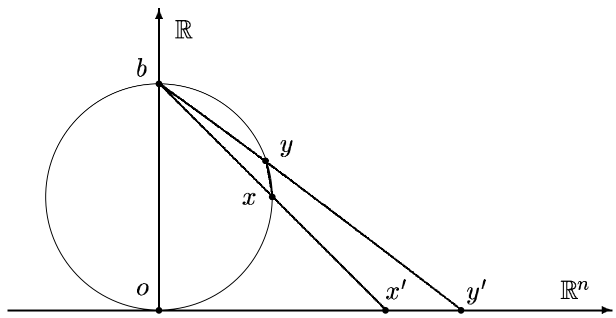

FIGURE 2. The standard stereographic projection.

First we will do a simple exercise in Euclidean geometry. Let x', y' be the respective stereographic projections of x, y (see Fig. 2). By the similarity of triangles obx and $x'bo$,

$$\frac{d_E(x,b)}{1} = \frac{1}{d_E(x',b)}, \qquad \text{i.e. } \mathrm{d}_E(x,b)\, d_E(x',b) = 1. \tag{23}$$

Similarly, $d_E(y,b)\, d_E(y',b) = 1$, hence

$$\frac{d_E(y',b)}{d_E(x,b)} = \frac{d_E(x',b)}{d_E(y,b)}.$$

Then triangles bxy and $by'x'$ are similar so

$$\frac{d_E(y',b)}{d_E(x,b)} = \frac{d_E(x',b)}{d_E(y,b)} = \frac{d_E(x',y')}{d_E(x,y)},$$

and using (23),

$$d_E(x',y') = \frac{d_E(x,y)\, d_E(y',b)}{d(x,b)} = \frac{d_E(x,y)}{d_E(x,b)\, d_E(y,b)}.$$

The projected metric d'_E on $\mathbb{S}^n$ was defined by $d'_E(x,y) := d_E(x',y')$, which is equivalent to

$$d'_E(x,y) := \frac{d_E(x,y)}{d_E(x,b)\, d_E(y,b)} \qquad \text{for } x,y \in \mathbb{S}^n \setminus \{b\}. \tag{24}$$

9.2. **Stereographic projection for hyperbolic complexes.** Let X be a hyperbolic complex. For $a \in X$, the metric $\check{d}_a$ on ∂X from section 3 will play the role of chordal metric.

Let $b \in \bar{X}$ and $a \in X$. In analogy with (24) we define a function

$$\check{d}_{a|b} : (X \cup (\partial X \setminus \{b\}))^2 \to [0, \infty) \quad \text{by}$$

$$\check{d}_{a|b}(x,y) := \frac{\check{d}_a(x,y)}{\check{d}_a(x,b)\, \check{d}_a(y,b)}. \tag{25}$$

Note that we consider a larger domain than in (8). One checks that the denominator never takes value 0 in this domain. We will say that $\check{d}_{a|b}$ is the *stereographic projection* of $\check{d}_a$ with respect to b. Also, $\check{d}_{a|b}(x,y)$ is a continuous function of four variables in the domain

$$\{(a,b,x,y) \mid a \in X,\ b \in \bar{X},\ x,y \in X \cup (\partial X \setminus \{b\})\}.$$

Now restrict $\check{d}_{a|b}$ to $(\partial X \setminus \{b\})^2$.

Theorem 16. *For any hyperbolic complex X, the function $\check{d}_{a|b}$ is a metric on $\partial X \setminus \{b\}$ conformally equivalent to $\check{d}_a$. The metrics $\check{d}_a$ and $\check{d}_{a|b}$ induce the same (usual) topology on $\partial X \setminus \{b\}$.*

Proof. x and y will always represent arbitrary points in $\partial X \setminus \{b\}$. Assume first that $b \in X$. By (25) and (6),

$$\begin{aligned}
\check{d}_{a|b}(x,y) &= \frac{e^{-\epsilon\langle x|y\rangle_a}}{e^{-\epsilon\langle b|x\rangle_a}\, e^{-\epsilon\langle b|y\rangle_a}} \\
&= e^{-\frac{\epsilon}{2}\left(\hat{d}(a,x)+\hat{d}(a,y)-\hat{d}(x,y)\right)+\frac{\epsilon}{2}\left(\hat{d}(a,b)+\hat{d}(a,x)-\hat{d}(b,x)\right)+\frac{\epsilon}{2}\left(\hat{d}(a,b)+\hat{d}(a,y)+\hat{d}(b,y)\right)} \\
&= e^{\epsilon\hat{d}(a,b)}\, e^{-\epsilon\langle x|y\rangle_b} = e^{\epsilon\hat{d}(a,b)}\, \check{d}_b(x,y).
\end{aligned} \tag{26}$$

Since $e^{\epsilon\hat{d}(a,b)}$ is a positive number, the triangle inequality for $\check{d}_b$ and (26) imply the triangle inequality for $\check{d}_{a|b}$. Then by continuity $\check{d}_{a|b}$ satisfies the triangle inequality when $b \in \bar{X}$ (in the domain $\partial X \setminus \{b\}$). It is obvious from (25) that $\check{d}_{a|b}(x,y) = 0$ if and only if $x = y$, so $\check{d}_{a|b}$ is a metric.

Now we show conformal equivalence. Assume first that $b \in X$. By (26) and Theorem 3,

$$\begin{aligned}
\frac{\check{d}_{a|b}(x,y)}{\check{d}_a(x,y)} &= \frac{e^{\epsilon\hat{d}(a,b)}\, e^{-\epsilon\langle x|y\rangle_b}}{e^{-\epsilon\langle x|y\rangle_a}} = e^{\epsilon(\hat{d}(a,b)+\langle x|y\rangle_a-\langle x|y\rangle_b)} \\
&= e^{\epsilon(\langle x|b\rangle_a+\langle y|b\rangle_a)} \in (0,\infty)
\end{aligned}$$

and by continuity this extends to the case when $b \in \bar{X}$ and $x, y \in \partial X \setminus \{b\}$. Then as $y \to x$ along $\partial X \setminus \{x\}$,

$$\lim \frac{\check{d}_{a|b}(x,y)}{\check{d}_a(x,y)} = \lim e^{\epsilon(\langle x|b\rangle_a + \langle y|b\rangle_a)} = e^{2\epsilon\langle x|b\rangle_a} \in (0,\infty).$$

$\check{d}_a$ and $\check{d}_{a|b}$ induce the same topology on $\partial X \setminus \{b\}$ because by the definition of $\check{d}_{a|b}$ they are bi-Lipschitz equivalent away from any open $\check{d}_a$-ball centered at b. □

A *horosphere* in X centered at u is the equivalence class in X where

$$x \sim y \qquad \Leftrightarrow \qquad \hat{\beta}_u(x,y) = 0.$$

We will understand horospheres in generalized sense: u will be a point in $\bar{X}$, not necessarily in ∂X. For $u \in X$, a horosphere centered at u is a usual metric sphere with respect to $\hat{d}$.

Lemma 17. *Let $b \in \bar{X}$ and $a, a' \in X$, then*

$$\check{d}_{a|b}(x,y) = e^{\epsilon\hat{\beta}_b(a,a')}\check{d}_{a'|b}(x,y).$$

In particular, if a and a' belong to the same horosphere in X centered at b, then $\check{d}_{a|b} = \check{d}_{a'|b}$.

Proof. If $b \in X$, by direct calculation as in (26),

$$\begin{aligned}\check{d}_{a|b}(x,y) &= e^{\epsilon\hat{d}(a,b)}e^{-\epsilon\langle x|y\rangle_b} = e^{\epsilon(\hat{d}(a,b)-\hat{d}(a',b))}e^{\epsilon\hat{d}(a',b)}e^{-\epsilon\langle x|y\rangle_b}\\ &= e^{\epsilon\hat{\beta}_b(a,a')}\check{d}_{a'|b}(x,y).\end{aligned}$$

This extends by continuity to $b \in \bar{X}$. □

Remark. For a CAT(-1) space X and $b \in \partial X$, Hersonsky and Paulin described a metric $d_{b,\mathcal{H}}$ [18]. Lemma 17 shows that $\check{d}_{a|b}$ is a generalization of $d_{b,\mathcal{H}}$ to arbitrary hyperbolic complexes, where $\mathcal{H}$ is the horosphere at b containing a. Also, $\check{d}_{a|b}$ improves the quasiconformal metric of [14, Ch.7, Prop.14].

The following is a description of the cross-ratio on ∂X in terms of $\check{d}_{a|b}$.

Proposition 18. *If $a \in X$, $(x, x', y, y') \in \bar{X}^\diamond$, and $b \in X \cup (\partial X \setminus \{x, x', y, y'\})$, then*

$$[\![x, x'|y, y']\!] = \frac{\check{d}_{a|b}(x,y)\ \check{d}_{a|b}(x',y')}{\check{d}_{a|b}(x,y')\ \check{d}_{a|b}(x',y)}.$$

In particular, the inclusion map $(\partial X \setminus \{b\}, \check{d}_{a|b}) \to (\partial X, \check{d}_a)$ preserves cross-ratio.

Proof. First suppose that $b \in X$. As in (26) wee obtain $\check{d}_{a|b}(x,y) = e^{\epsilon \hat{d}(a,b)} d_b(x,y)$. This and Proposition 9 imply the required equality, which extends by continuity to the case $b \in \partial X \setminus \{x, x', y, y'\}$. □

10. Hyperbolic dimension.

In this section we define a notion of hyperbolic dimension for groups and spaces. This takes it origins in and is related to the notion of conformal dimension; the reader is advised to use the following references as a guide: Margulis [19], Pansu [24], Gromov [16, 15], Bonk-Kleiner [2], [1], [3].

10.1. **Hyperbolic spaces.** Given a number $C \in [0, \infty)$ and a metric space (X, d), a C^+*geodesic* in X is a (not necessarily continuous) map $\gamma : [0, T] \to X$ such that $\big|d(\gamma(s), \gamma(t)) - |s - t|\big| \le C$ for all $s, t \in [0, T]$. A metric space X is called $^+$*geodesic* if there is $C \in [0, \infty)$ such that every two points in X can be connected by a C^+geodesic.

We will work in the category of $^+$geodesic metric spaces. The notion of hyperbolicity can be defined for $^+$geodesic metric spaces, for example using $^+$*geodesic* thin triangles. In what follows, hyperbolic spaces can be just as well assumed to be hyperbolic in the above generalized sense. A metric space X is *proper* if closed balls in X are compact.

10.2. **Two classes of metrics** $\hat{\mathrm{M}}$ **and** $\check{\mathrm{M}}$**.** Suppose (X, d) is a proper hyperbolic space and $\hat{d}$ is another metric on X. Let $\langle \cdot | \cdot \rangle_{\cdot}$ be the Gromov product defined by $\hat{d}$ and denote

$$E_{\hat{d}} := \{\epsilon \in (0, \infty) \mid \forall a \in X \;\; e^{-\epsilon \langle \cdot | \cdot \rangle_a} \text{ restricted to } (\partial X)^2 \text{ is a metric}\},$$

$$\epsilon_{\hat{d}} := \begin{cases} \sup E_{\hat{d}} & \text{if } E_{\hat{d}} \ne \emptyset, \\ 0 & \text{if } E_{\hat{d}} = \emptyset. \end{cases}$$

For $a \in X$ and $\epsilon \in (0, \infty)$ define a function $\check{\ }_{a,\epsilon}(\hat{d}) : X^2 \to [0, 1]$ by

$$\check{\ }_{a,\epsilon}(\hat{d})(x, y) := e^{-\epsilon \langle x | y \rangle_a}, \qquad x, y \in X, \tag{27}$$

where again $\langle \cdot | \cdot \rangle_{\cdot}$ is defined by $\hat{d}$.

Now additionally assume that (X, d) is given an isometric action by a group Γ. Denote $\hat{\mathrm{M}}(X, \Gamma)$ the set of all metrics $\hat{d}$ on X satisfying the following.

(a) $\hat{d}$ is quasiisometric to d.
(b) $\hat{d}$ is Γ-invariant.
(c) $(X, \hat{d})$ is $^+$geodesic.

(d) The $\hat{d}$-double difference $\langle\cdot,\cdot|\cdot,\cdot\rangle$ on X^4 extends to a continuous function $\bar{X}^\diamond \to [-\infty,\infty]$. In particular, for all $\epsilon \in (0,\infty)$, $\check{}_{a,\epsilon}(\hat{d})$ extends to a continuous function $\check{}_{a,\epsilon}(\hat{d}) : \bar{X}^2 \to [0,1]$.

(e) $1 \in E_{\hat{d}}$, i.e. $e^{-\langle\cdot|\cdot\rangle_a}$ is a metric on ∂X for all $a \in X$.

Using Theorem 5 and rescaling $\hat{d}$ if needed to guarantee (e), we see that $\hat{\mathrm{M}}(X,\Gamma)$ is non-empty when X is a hyperbolic complex and Γ is a group of its isometries. Further denote $\hat{\mathrm{M}}(X) := \hat{\mathrm{M}}(X,1)$; this is the corresponding class of metrics with no equivariance requirement.

The second class of metrics is on the ideal boundary of X.

$$\begin{aligned}
&\check{\mathrm{M}}_a(X,\Gamma) := \{\check{}_{a,\epsilon}(\hat{d}) \mid \hat{d} \in \hat{\mathrm{M}}(X,\Gamma),\ \epsilon \in E_{\hat{d}}\},\\
&\check{\mathrm{M}}(X,\Gamma) := \bigcup_{a\in X} \check{\mathrm{M}}_a(X,\Gamma),\\
&\check{\mathrm{M}}_a(X) := \check{\mathrm{M}}_a(X,1), \qquad \check{\mathrm{M}}(X) := \check{\mathrm{M}}(X,1).
\end{aligned}$$

$\check{\mathrm{M}}(X)$ consists of metrics on ∂X.

Lemma 19. *If $\hat{d} \in \hat{\mathrm{M}}(X)$, $a \in X$ and $\epsilon \in (0,\infty)$, then $\check{}_{a,\epsilon}(\hat{d})$ induces the usual topology on ∂X.*

The proof is the same as for geodesic metric spaces (see for example [14, Ch.7, Prop.14]). For the statement to hold, $\check{}_{a,\epsilon}(\hat{d})$ does not need to be a metric; only conditions (a), (c) and (d) suffice.

10.3. **The definition of hyperbolic dimension.** Hdim will stand for Hausdorff dimension. Given a proper hyperbolic metric space X with a Γ-action, the *hyperbolic dimension of* (X,Γ) is the quantity

$$\begin{aligned}
\hbar(X,\Gamma) &:= \inf\{\mathrm{Hdim}\big(\partial X, \check{}_{a,\epsilon}(\hat{d})\big) \mid \hat{d} \in \hat{\mathrm{M}}(X,\Gamma),\ a \in X,\ \epsilon \in E_{\hat{d}}\}\\
&= \inf\{\mathrm{Hdim}(\partial X, \check{d}) \mid \check{d} \in \check{\mathrm{M}}(X,\Gamma)\}.
\end{aligned}$$

For a hyperbolic group Γ, the *hyperbolic dimension of* Γ is

$$\hbar(\Gamma) := \hbar(\Gamma,\Gamma),$$

where Γ is viewed as a hyperbolic metric space with the left Γ-action.

We set by definition $\mathrm{Hdim}(\emptyset) := -1$, so $\hbar(\Gamma) = -1$ for finite groups.

Remark. The above definition allows for generalizations. One could work with pseudometrics instead of metrics both in X and in ∂X. The Γ-action on X can be also replaced with a $^+$isometric action, or with an isometric $^+$action, which are the corresponding notions defined up to a uniform additive constant.

10.4. **Some properties.**

Proposition 20. *Suppose X is a proper hyperbolic space such that ∂X is not totally disconnected and let $\hat{d} \in \hat{\mathrm{M}}(X)$. Then*

(1) $\epsilon_{\hat{d}} \in (0, \infty)$ *and*
(2) *for all $a \in X$, the restriction of $e^{-\epsilon_{\hat{d}}\langle\cdot|\cdot\rangle_a}$ to $(\partial X)^2$ is a metric on ∂X.*

Proof. (**1**) By the definition of $\hat{\mathrm{M}}(X)$, $\epsilon_{\hat{d}} > 0$ (actually $\epsilon_{\hat{d}} \geq 1$). Now we show $\epsilon_{\hat{d}} < \infty$.

Fix any $a \in X$ and choose $\epsilon \in (0, \infty)$ so that $e^{-\epsilon\langle\cdot|\cdot\rangle_a}$ is a metric on ∂X. Pick a connected component C of ∂X with more than one point. C is closed in ∂X, therefore compact. Since $e^{-\epsilon\langle\cdot|\cdot\rangle_a}$ induces the topology of ∂X, it is continuous with respect to this topology. Then there exist two distinct points $x, y \in C$ such that

$$e^{-\epsilon\langle x|y\rangle_a} = \mathrm{diam}(C) := \sup\{e^{-\epsilon\langle x'|y'\rangle_a} \mid x', y' \in C\} > 0.$$

Let $C_x := \{z \in C \mid e^{-\epsilon\langle z|y\rangle_a} = e^{-\epsilon\langle x|y\rangle_a}\}$. Since $e^{-\epsilon\langle\cdot|\cdot\rangle_a}$ is continuous, C_x is closed in ∂X. Since C is closed, the closure of $C \setminus C_x$ in X, $\overline{C \setminus C_x}$, lies in C. Since C is connected, $C \setminus C_x$ cannot be closed in C, so

$$(\overline{C \setminus C_x}) \cap C_x = (\overline{C \setminus C_x}) \setminus (C \setminus C_x) \neq \emptyset.$$

Take any $z \in (\overline{C \setminus C_x}) \cap C_x$, then we can choose $u \in C \setminus C_x$ sufficiently close to z so that $e^{-\epsilon\langle z|u\rangle_a} < e^{-\epsilon\langle x|y\rangle_a}$. Since $u \in C \setminus C_x$, we also have $e^{-\epsilon\langle u|y\rangle_a} < e^{-\epsilon\langle x|y\rangle_a}$. The last two inequalities imply that there exists $\alpha_0 \in (0, \infty)$ such that for all $\alpha \geq \alpha_0$,

$$\left(\frac{e^{-\epsilon\langle z|u\rangle_a}}{e^{-\epsilon\langle x|y\rangle_a}}\right)^\alpha + \left(\frac{e^{-\epsilon\langle u|y\rangle_a}}{e^{-\epsilon\langle x|y\rangle_a}}\right)^\alpha < 1,$$

or equivalently,

$$e^{-\alpha\epsilon\langle z|u\rangle_a} + e^{-\alpha\epsilon\langle u|y\rangle_a} < e^{-\alpha\epsilon\langle x|y\rangle_a}.$$

Thus for each $\alpha \geq \alpha_0$, $e^{-\alpha\epsilon\langle\cdot|\cdot\rangle_a}$ is not a metric, so $\epsilon_d \leq \alpha_0\epsilon < \infty$.
(**2**) The triangle inequality for $e^{-\epsilon_{\hat{d}}\langle\cdot|\cdot\rangle_a}$ follows by continuity as $\epsilon \nearrow \epsilon_{\hat{d}}$. Also since $\epsilon_{\hat{d}} > 0$ and $(X, \hat{d})$ is $^+$geodesic,

$$e^{-\epsilon_{\hat{d}}\langle x|y\rangle_a} = 0 \quad \Leftrightarrow \quad \langle x|y\rangle_a = \infty \quad \Leftrightarrow \quad x = y.$$

□

Hyperbolic groups with totally disconnected boundary are completely described by the following theorem. The proof using the Dunwoody's accessibility theorem [11] can be deduced from [14, Ch.7,Th.19].

Theorem 21. *The ideal boundary of a hyperbolic group* Γ *is totally disconnected iff* Γ *contains a free subgroup of finite index.*

By definition, totally disconnected spaces can be empty, so the statement includes the case when Γ is finite (virtually free of rank 0), and also when Γ is virtually $\mathbb{Z}$ (virtually free of rank 1). For a topological space Z, $\dim(Z) = 0$ iff Z is totally disconnected and non-empty. Here $dim(Z)$ denotes the topological dimension of Z.

Theorem 22. *The following are satisfied for a hyperbolic group* Γ.

(a) $\hbar(\Gamma) = -1$ *iff* Γ *is finite.*
(b) $\hbar(\Gamma) = 0$ *iff* Γ *contains a finite index free subgroup of rank* ≥ 1.

Proof. **(a)** holds because Γ is finite iff $\partial\Gamma$ is empty.
(b) First assume that $\Gamma = F$ is a free finitely generated group. The Cayley graph of F with respect to a free basis is a tree T. Let d be the word metric on the tree and $\check{d}(x, y) := e^{-\epsilon\langle x|y\rangle_a}$. Since T is a tree,

$$\langle x|z\rangle_a \geq \min\{\langle x|y\rangle_a, \langle y|z\rangle_a\}$$

holds for $x, y, z \in T$, and by continuity for $x, y, z \in \bar{T}$, we have

$$e^{-\epsilon\langle x|z\rangle_a} \leq \max\{e^{-\epsilon\langle x|y\rangle_a}, e^{-\epsilon\langle y|z\rangle_a}\}$$

for all $x, y, z \in \partial T$. This implies that $\check{}_{a,\epsilon}(d) = e^{-\epsilon\langle\cdot|\cdot\rangle_a} = \check{d}^{\epsilon}$ is a metric on ∂X for all $\epsilon \in (0, \infty)$. One checks that this metric is in $\hat{M}(X, \Gamma)$. But $\mathrm{Hdim}\big(\partial X, \check{d}^{\epsilon}\big) \to 0$ as $\epsilon \to \infty$, so $\hbar(\Gamma) = 0$.

In the general case when Γ contains a free group of finite index, Γ acts on a tree T with finite stabilizers of vertices ([25, Theorem 7.3]). Denote d_T the path metric in T and fix a vertex v in T. Given two distinct elements $g, h \in \Gamma$, let $d(g, h) := d_T(gv, hv)$ if $gv \neq hv$, and $d(g, h) := 1$ if $gv = hv$. This defines a Γ-invariant metric d on Γ that behaves just like the metric d_T on T. The metrics d and d_T induce the same metric $\check{d}$ on $\partial\Gamma = \partial T$, so by rescaling d as above we obtain $\hbar(\Gamma) = 0$.

Conversely, if $\hbar(\Gamma) = 0$, then by definition Γ is infinite, and since the topological dimension $dim(\partial X)$ is at most the Hausdorff dimension of ∂X, we have

$$dim(\partial X) \leq \hbar(\Gamma) = 0,$$

hence $dim(\partial X) = 0$. Now by Theorem 21, Γ is virtually free, and by (a), it is virtually free of rank at least one. □

Corollary 23. *For a hyperbolic group* Γ, *if* $0 \leq \hbar(\Gamma) < 1$ *then* $\hbar(\Gamma) = 0$.

From now on we will additionally assume that ∂X is not totally disconnected. Denote

$$\hat{\mathrm{M}}_1(X,\Gamma) := \{\hat{d} \in \hat{\mathrm{M}}(X,\Gamma) \mid \epsilon_{\hat{d}} = 1\},$$
$$\hat{\mathrm{M}}_1(X) := \hat{\mathrm{M}}_1(X,1).$$

The space $\hat{\mathrm{M}}_1(X,\Gamma)$ is non-empty because by Proposition 20 for any metric $\hat{d} \in \hat{\mathrm{M}}(X,\Gamma)$, $\epsilon_{\hat{d}}\hat{d} \in \hat{\mathrm{M}}_1(X,\Gamma)$.

Definition 24. *Given $\hat{d} \in \hat{\mathrm{M}}(X)$, let*

$$\check{}_a(\hat{d})(x,y) := e^{-\epsilon_{\hat{d}}\langle x|y\rangle_a}, \qquad a \in X,\ x,y \in \bar{X},$$

where $\langle\cdot|\cdot\rangle_.$ is the Gromov product defined by $\hat{d}$.

In particular, if $\hat{d} \in \hat{\mathrm{M}}_1(X)$, then

$$\check{}_a(\hat{d})(x,y) = e^{-\langle x|y\rangle_a}, \qquad a \in X,\ x,y \in \bar{X}.$$

Note that $\check{}_a$ is scale-invariant, i.e. $\check{}_a(\hat{d}) = \check{}_a(c\hat{d})$ for any $c \in (0,1)$. This also holds for any $c \in (0,\infty)$ as long as we do not require the metric $c\hat{d}$ to be in $\hat{\mathrm{M}}(X)$.

In view of Proposition 20, when ∂X is not totally disconnected, the hyperbolic dimension satisfies

$$\begin{aligned}\hbar(X,\Gamma) &= \inf\{\mathrm{Hdim}\big(\partial X, \check{}_a(\hat{d})\big) \mid \hat{d} \in \hat{\mathrm{M}}_1(X,\Gamma),\ a \in X\} \\ &= \inf\{\mathrm{Hdim}\big(\partial X, \check{}_a(\hat{d})\big) \mid \hat{d} \in \hat{\mathrm{M}}(X,\Gamma),\ a \in X\}.\end{aligned}$$

11. Equivariant structures.

The following theorem makes a case for the use of Γ-equivariance in the definition of hyperbolic dimension.

Theorem 25. *Let Γ be a group, X and X' be hyperbolic complexes with properly discontinuous cocompact Γ-actions by isometries (= simplicial automorphisms), $a \in X$, $a' \in X'$, $\check{d}_a \in \check{\mathrm{M}}_a(X,\Gamma)$, $\check{d}_{a'} \in \check{\mathrm{M}}_{a'}(X',\Gamma)$. If $f : (\partial X, \check{d}_a) \to (\partial X', \check{d}_{a'})$ is a homeomorphism commuting with the Γ-actions on ∂X and $\partial X'$, then the following statements are equivalent.*

1. *f is conformal in ∂X.*
2. *f is symmetric in ∂X.*
3. *f preserves cross ratio, i.e. $[\![f(b), f(c)|f(x), f(y)]\!] = [\![b,c|x,y]\!]$ for all $(b,c,x,y) \in \partial^\diamond X$.*
4. *For each pairwise distinct triple $x,u,v \in \partial X$,*

$$\lim \frac{[\![f(x), f(u)|f(y), f(v)]\!]}{[\![x,u|y,v]\!]} \quad \text{as} \quad y \to x \quad \text{along } \partial X \setminus \{x\}$$

exists and equals 1.

(5) *For each pairwise distinct triple* $b, u, v \in \partial X$,

$$\lim \frac{[\![f(b), f(u)|f(y), f(v)]\!]}{[\![x, u|y, v]\!]} \quad \textit{as} \quad y \to b \quad \textit{along } \partial X \setminus \{b\}$$

exists in $(0, \infty)$.

(6) *The metric derivative* $|f'(x)|$ *is well-defined in* $(0, \infty)$ *at each* $x \in \partial X$*, it is continuous as a function of* x*, and the restricted function*

$$f : (\partial X \setminus \{x\}, \check{d}_{a|x}) \to (\partial X \setminus \{x'\}, \check{d}_{a'|f(x)})$$

is a similarity with factor $1/|f'(x)|$.

Proof. If X is elementary, i.e. ∂X consists of at most 2 points, the equivalence of the above statements is a tedious triviality. From now on we will assume that ∂X consists of more than 2 points, so in particular it does not have isolated points.

(**6**) $\Rightarrow$ (**1**) and (**3**) $\Rightarrow$ (**4**) $\Rightarrow$ (**5**) are obvious.

(**1**) $\Rightarrow$ (**2**) It follows from Definition 11 that conformal maps are symmetric.

(**2**) $\Rightarrow$ (**3**) Let $b, c \in \partial X$ be, respectively, the repelling and attracting points of a hyperbolic isometry $g \in \Gamma$. Since f commutes with g, the points $b' := f(b)$ and $c' := f(c)$ in $\partial X'$ are fixed by g and are, respectively, repelling and attracting for g.

Let $x, y \in \partial X \setminus \{b, c\}$ and denote for simplicity

$$\begin{aligned}
&x' := f(x), \quad x_i := g^i(x),\\
&x'_i := g^i(x') = g^i(f(x)) = f(g^i(x)) = f(x_i),\\
&y' := f(y), \quad y_i := g^i(y),\\
&y'_i := g^i(y') = g^i(f(y)) = f(g^i(y)) = f(y_i).
\end{aligned}$$

Since g is an isometry both in X and in X', it preserves the cross-ratios, and since it fixes b, c, b', c', we have

$$[\![b, c|x, y]\!] = [\![b, c|x_i, y_i]\!] \quad \text{and} \quad [\![b', c'|x', y']\!] = [\![b', c'|x'_i, y'_i]\!] \tag{28}$$

for all i.

Using (28), Theorem 8, (20) and the assumption that f is symmetric, we obtain

$$
\begin{aligned}
\frac{[\![b',c'|x',y']\!]}{[\![b,c|x,y]\!]} &= \lim_{i\to\infty}\frac{[\![b',c'|x'_i,y'_i]\!]}{[\![b,c|x_i,y_i]\!]} = \lim_{i\to\infty}\frac{[\![b',a'|x'_i,y'_i]\!]\,[\![a',c'|x'_i,y'_i]\!]}{[\![b,a|x_i,y_i]\!]\,[\![a,c|x_i,y_i]\!]} \\
&= \lim_{i\to\infty}\frac{[\![b',a'|x'_i,y'_i]\!]}{[\![b_k,a|x_i,y_i]\!]}\cdot\lim_{i\to\infty}\frac{\check{d}_{a'}(c',y'_i)\ \check{d}_a(c,x_i)}{\check{d}_{a'}(c',x'_i)\ \check{d}_a(c,y_i)} \\
&= \frac{[\![b',a'|c',c']\!]}{[\![b,a|c,c]\!]}\cdot\lim_{i\to\infty}\frac{\check{d}_{a'}(f(c),f(y_i))\ \check{d}_a(c,x_i)}{\check{d}_{a'}(f(c),f(x_i))\ \check{d}_a(c,y_i)} = \frac{1}{1}\cdot 1 = 1.
\end{aligned}
$$

This shows that the cross-ratio $[\![b,c|x,y]\!]$ is preserved by f provided b and c are fixed points of a hyperbolic isometry $g\in\Gamma$. But such pairs are dense in $(\partial X)^2$ ([15, Corollary 8.2.G]), therefore by the continuity of f, the cross ratio is preserved by f in the whole domain $\partial^{\diamond}X$.

(4) $\Rightarrow$ **(6)** Fix an arbitrary $x\in\partial X$ and then choose $u,v\in\partial X$ so that x,u,v are pairwise distinct. Take $y\in\partial X\setminus\{x\}$ sufficiently close to x so that $y\in\partial X\setminus\{x,u,v\}$. Denote u',v',x',y' the respective images of u,v,x,y under f. As $y\to x$ along $\partial X\setminus\{x\}$,

$$
\begin{aligned}
|f'(x)| &= \lim\frac{\check{d}_{a'}(f(x),f(y))}{\check{d}_a(x,y)} = \lim\frac{\check{d}_{a'}(x',y')}{\check{d}_a(x,y)} = \lim\frac{e^{-\epsilon\langle x'|y'\rangle_{a'}}}{e^{-\epsilon\langle x|y\rangle_a}} \\
&= \lim\frac{e^{\epsilon'\langle x',u'|y',v'\rangle-\epsilon'\langle x'|v'\rangle_{a'}-\epsilon'\langle y'|u'\rangle_{a'}+\epsilon'\langle u'|v'\rangle_{a'}}}{e^{\epsilon\langle x,u|y,v\rangle-\epsilon\langle x|v\rangle_a-\epsilon\langle y|u\rangle_a+\epsilon\langle u|v\rangle_a}} \\
&= \lim\frac{[\![x',u'|y',v']\!]}{[\![x,u|y,v]\!]}\cdot\lim\frac{e^{-\epsilon'\langle x'|v'\rangle_{a'}-\epsilon'\langle y'|u'\rangle_{a'}+\epsilon'\langle u'|v'\rangle_{a'}}}{e^{-\epsilon\langle x|v\rangle_a-\epsilon\langle y|u\rangle_a+\epsilon\langle u|v\rangle_a}} \\
&= \lim\frac{[\![x',u'|y',v']\!]}{[\![x,u|y,v]\!]}\cdot\frac{e^{-\epsilon'\langle x'|v'\rangle_{a'}}\ e^{-\epsilon'\langle x'|u'\rangle_{a'}}\ e^{\epsilon'\langle u'|v'\rangle_{a'}}}{e^{-\epsilon\langle x|v\rangle_a}\ e^{-\epsilon\langle x|u\rangle_a}\ e^{\epsilon\langle u|v\rangle_a}} \\
&= 1\cdot\frac{\check{d}_a(u,v)}{\check{d}_a(x,u)\,\check{d}_a(x,v)}\frac{\check{d}_{a'}(x',u')\,\check{d}_{a'}(x',v')}{\check{d}_{a'}(u',v')} \\
&= \frac{\check{d}_{a|x}(u,v)}{\check{d}_{a'|x'}(u',v')}\in(0,\infty).
\end{aligned}
$$

All the statements of (6) follow from the above equality.

(5) $\Rightarrow$ **(1)** is similar to (4) $\Rightarrow$ (6): as $y \to b$ along $\partial X \setminus \{b\}$,

$$|f'(b)| = \lim \frac{[\![b', u'|y', v']\!]}{[\![b, u|y, v]\!]} \cdot \lim \frac{e^{-\epsilon'\langle b'|v'\rangle_{a'} - \epsilon'\langle y'|u'\rangle_{a'} + \epsilon'\langle u'|v'\rangle_{a'}}}{e^{-\epsilon\langle b|v\rangle_a - \epsilon\langle y|u\rangle_a + \epsilon\langle u|v\rangle_a}}$$

$$= \lim \frac{[\![b', u'|y', v']\!]}{[\![b, u|y, v]\!]} \cdot \frac{e^{-\epsilon'\langle b'|v'\rangle_{a'}} \, e^{-\epsilon'\langle b'|u'\rangle_{a'}} \, e^{\epsilon'\langle u'|v'\rangle_{a'}}}{e^{-\epsilon\langle b|v\rangle_a} \, e^{-\epsilon\langle b|u\rangle_a} \, e^{\epsilon\langle u|v\rangle_a}}$$

$$= \lim \frac{[\![b', u'|y', v']\!]}{[\![b, u|y, v]\!]} \cdot \frac{\check{d}_a(u, v)}{\check{d}_a(b, u)\, \check{d}_a(b, v)} \frac{\check{d}_{a'}(b', u')\, \check{d}_{a'}(b', v')}{\check{d}_{a'}(u', v')}$$

$$= \lim \frac{[\![b', u'|y', v']\!]}{[\![b, u|y, v]\!]} \cdot \frac{\check{d}_{a|b}(u, v)}{\check{d}_{a'|b'}(u', v')} \in (0, \infty).$$

□

Remark. In [27] and [29], Tukia considered an equivariant map f between subsets of $\mathbb{S}^n$, and, both in differentiable and measurable settings, provides sufficient conditions for it to be Möbius. In our setting, Γ can be any hyperbolic group and the derivative is the metric one.

12. Questions.

Any open questions about the Pansu's conformal dimension and its Ahlfors regular version can be asked about the hyperbolic dimension $\hbar(\Gamma)$ as well. We present questions; some of them were inspired by the work of Bonk and Kleiner [3].

Given a hyperbolic group Γ, it might happen that there exists $\hat{d} \in \hat{\mathrm{M}}(X, \Gamma)$ such that $\mathrm{Hdim}(\hat{d}) = \hbar(\Gamma)$. In this case we will say that the hyperbolic dimension $\hbar(\Gamma)$ is *achieved*, or *realized at* $\hat{d}$.

Question 1. *Under what assumptions on Γ and $\partial\Gamma$ is $\hbar(\Gamma)$ achieved? Equivalently, when there exists $\check{d} \in \check{\mathrm{M}}(\Gamma, \Gamma)$ such that $\mathrm{Hdim}\big(\partial\Gamma, \check{d}\big) = \hbar(\Gamma)$ for all $a \in X$? Is it achieved if $\partial\Gamma$ is homeomorphic to $\mathbb{S}^2$?*

This question is inspired by the Cannon's conjecture and results in [3].

Question 2. *Let Γ be a hyperbolic group with $\partial\Gamma$ homeomorphic to $\mathbb{S}^2$ and d be the chordal metric on $\mathbb{S}^2$. Does there exist $\alpha \in (0, 1]$ and a conformal homeomorphism $(\partial\Gamma, \check{d}_a) \to (\mathbb{S}^2, d^\alpha)$?*

Question 3. *More generally, if a group Γ acts topologically transitively by conformal homeomorphisms on a metric space $(Z, \check{d})$ homeomorphic to $(\mathbb{S}^2, d)$, does there exist $\alpha \in (0, 1]$ and a conformal homeomorphism $(Z, \check{d}) \to (\mathbb{S}^2, d^\alpha)$?*

Question 4. *If the answer to Question 2 is "no", is it "yes" under the additional assumption that the hyperbolic dimension of* Γ *is achieved? Furthermore, if* $\check{d} \in \check{\mathrm{M}}(\Gamma, \Gamma)$ *is such that* $\mathrm{Hdim}(\partial\Gamma, \check{d}) = \hbar(\Gamma)$*, does there exist a conformal homeomorphism* $(\partial\Gamma, \check{d}) \to (\mathbb{S}^2, d)$*?*

Questions 2-4 are analogs of the Riemann mapping theorem in the equivariant setting.

Following Bonk and Kleiner [3], define the Ahlfors regular conformal dimension of $\partial\Gamma$, denoted $\mathrm{Arcd}(\partial\Gamma)$, as the infimum of Hausdorff dimensions of Ahlfors regular metrics which are quasisymmetrically equivalent to $\check{d}$.

Question 5. *Is there an example of a group* Γ *for which* $\mathrm{Arcd}(\Gamma) \neq \hbar(\Gamma)$*?*

Question 6. *If* $\partial\Gamma$ *is homeomorphic to* $\mathbb{S}^2$*, then is* $\hbar(\Gamma)$ *achieved?*

Question 7. *Let* Γ *be a hyperbolic group with connected* $\partial\Gamma$*. Suppose that* $\hat{d}_1, \hat{d}_2 \in \hat{\mathrm{M}}_1(\Gamma, \Gamma)$ *are such that for any* $a \in \Gamma$*,*

$$\mathrm{Hdim}\big(\partial\Gamma, \check{\mathrm{a}}(\hat{\mathrm{d}}_1)\big) = \mathrm{Hdim}\big(\partial\Gamma, \check{\mathrm{a}}(\hat{\mathrm{d}}_2)\big) = \hbar(\Gamma) > 1.$$

Then are $\hat{d}_1$ *and* $\hat{d}_2$ $^{+}$*equivalent, i.e. is* $|\hat{d}_1 - \hat{d}_2|$ *bounded?*

If the answer is yes, this would be a version of the Mostow rigidity theorem in the discrete group setting. The condition $\hbar(\Gamma) > 1$ excludes the surface case.

Question 8. *Suppose* Γ *is a hyperbolic group such that* $\partial\Gamma$ *is homeomorphic to the* n*-dimensional sphere. Does it follow that* Γ *acts properly discontinuously and cocompactly by isometries on a simply connected Riemannian* $(n+1)$*-manifold? Does there exist such a manifold with negative sectional curvature?*

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