ℓ_∞-COHOMOLOGY AND METABOLICITY OF NEGATIVELY CURVED COMPLEXES.

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Abstract. We prove the analog of de Rham's theorem for ℓ_∞-cohomology of the universal cover of a finite simplicial complex. A sufficient criterion is given for linearity of isoperimetric functions for filling cycles of any positive dimension over $\mathbb{R}$. This implies the linear higher dimensional isoperimetric inequalities for the fundamental groups of finite negatively curved complexes and of closed negatively curved manifolds. Also, these groups are $\mathbb{R}$-metabolic.

1. Introduction.

This paper discusses topics related to ℓ_∞-cohomology: forms on simplicial complexes and metabolicity.

Various applications of forms were shown by H. Whitney in [11] and by P. A. Griffiths and J. W. Morgan in [8]. In particular, de Rham's theorem was proved for different types of cohomology. The second section of the present paper lists the necessary definitions and the third section gives a proof of de Rham's theorem for ℓ_∞-cohomology.

Also we are going to discuss the concept of metabolicity. The term was suggested by S. Gersten. A group is called *metabolic* or $\mathbb{Z}$*-metabolic* if $H^2_{(\infty)}(X, A) = 0$ for any normed abelian group A, where $H_{(\infty)}$ stands for the ℓ_∞-cohomology and X is a $K(G, 1)$ with finite 2-skeleton. Replacing normed abelian groups with normed vector spaces over $\mathbb{R}$ one obtains the definition of an $\mathbb{R}$*-metabolic* group. Obviously, "metabolic" implies "$\mathbb{R}$-metabolic". Also it can be deduced from the argument in [3] that "$\mathbb{R}$-metabolic" implies "hyperbolic", which justifies the choice of the funny term. It is an open question whether these three concepts are equivalent.

Metabolic groups are of particular interest because, for example, they satisfy the higher dimensional analog of the linear isoperimetric inequality as shown in [4]. They also admit a more transparent geometric definition. S. Gersten showed in [5] that the fundamental group of a closed Riemannian manifold of negative curvature is $\mathbb{R}$-metabolic and in [4] he suggested that it may be true for the fundamental groups of all finite negatively curved complexes. The fourth section of the present paper is devoted to proving $\mathbb{R}$-metabolicity in this case. This was suggested by D. Toledo that forms on complexes may be used here.

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In the fifth section we prove a sufficient criterion for a group to have a linear isoperimetric function for cycles of any positive dimension over $\mathbb{R}$. This will imply linear higher dimensional functions in the cases when G is (1) the fundamental group of a finite negatively curved complex or (2) the fundamental group of a closed negatively curved manifold. The latter case was shown in [7] by geometric methods.

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2. Differential forms on complexes and cohomology.

First we need several definitions. For $K \leq 0$, let M_K^n be the standard n-dimensional space of constant curvature K (so it is the Euclidean space $\mathbb{E}^n$ for $K = 0$ and the hyperbolic space $\mathbb{H}^n$ for $K = -1$).

Definition 2.1. *X is called an M_K-*complex *if*

(a) *X is a simplicial complex,*
(b) *each closed n-simplex Δ in X has an embedding $f : \Delta \to M_K^n$ onto a convex hull of $n+1$ points in M_K^n, so that Δ possesses the induced metric, and*
(c) *for each simplex Δ in X, and any face σ of Δ, the inclusion $\sigma \hookrightarrow \Delta$ is an isometric embedding.*

For the standard (closed) simplex Δ^n, there is an affine embedding $f_\Delta : \Delta^n \to \mathbb{E}^n$, so that the images of the 1-faces of Δ have length 1. This embedding is unique up to an isometry of $\mathbb{E}$. So any simplicial complex X can be thought of as an $\mathbb{E}$-complex by taking the embeddings $f_\Delta : D^n \to \mathbb{E}^n$. Obviously, the conditions (a),(b), and (c) are satisfied.

Definition 2.2. *A constant appearing in the discussion below will be called* universal *if it depends only on the triangulation of X.*

Let Δ be a (closed) simplex in an M_K-complex X and σ be a face of Δ. Viewing Δ as embedded into M_K^n by f_Δ, there is a unique hyperplane P in M_K^n with $P \cap f_\Delta(\Delta) = f_\Delta(\sigma)$, so there is a unique isometry $M_K^{n-1} \to P$ taking $f_\sigma(\sigma)$ onto $f_\Delta(\sigma)$.

By an abuse of notation, we will usually identify each simplex Δ in X with its image $f_\Delta(\Delta)$ in M_K^n. Viewing each simplex Δ of X as embedded into the standard space enables us to talk about the tangent space to Δ. By that we mean the restriction of a tangent space from a neighborhood of Δ. We denote it by $T\Delta$. Also, the discussion above implies that these tangent spaces are compatible in the sense that if σ is a face of Δ, then the inclusion $\sigma \hookrightarrow \Delta$ induces an inclusion (= injective bundle map) $T\sigma \hookrightarrow T\Delta$, which is, by definition, the restriction of the bundle map induced by the immersion $M_K^{n-1} \to P$.

An *r-form* on Δ^n is the restriction to Δ of a smooth r-form in an open neighborhood of Δ in M_K^n. If σ is a face of Δ and ω_Δ and ω_σ are forms on Δ and σ, respectively, we say that ω_σ *is compatible with* ω_Δ if ω_σ is the pull-back of ω_Δ under the inclusion $\sigma \hookrightarrow \Delta$.

Definition 2.3. *A* form *ω on a M_K-*complex *X is a collection of forms ω_Δ, one for each simplex Δ in X, such that ω_σ is compatible with ω_Δ whenever σ is a face of Δ.*

Given a form $\omega = \{\omega_\Delta\}$ on X, the differentials $d\omega_\Delta$ are well defined forms on simplices Δ. Also, if $i : \sigma \hookrightarrow \Delta$ is an inclusion of a face σ to Δ, then $d\omega_\sigma = (d \circ i^*)(\omega_\Delta) = (i^* \circ d)(\omega_\Delta) = i^*(d\omega_\Delta)$. It shows that the collection $\{d\omega_\Delta\}$ is compatible under restrictions to faces, so it gives a form on X, which we denote by $d\omega$ and call *the differential* of ω.

All the above definitions work in a more general setting, namely for forms with values in an arbitrary Banach space V. We describe them as follows.

For a finite dimensional space L with basis $x_1, ..., x_n$, $L_{[r]}$ denotes the quotient of the r-fold tensor product $L^{\otimes r}$ by the alternating relations $x_1 \otimes ... \otimes x_r = 0$ whenever $x_i = x_j$ for some $1 \le i < j \le r$. $x_1 \wedge ... \wedge x_r$ denotes an element of $L_{[r]}$, $x_i \in L$, $1 \le i \le r$. The elements of $L_{[r]}$ are called *r-multivectors* (*skew-symmetric tensors* is another name). $\{e_\lambda \mid \lambda = (i_1, ..., i_r), i_1 < ... < i_r\}$ is a standard basis of $L_{[r]}$, where $e_\lambda := x_{i_1} \wedge ... \wedge x_{i_r}$. A scalar product on L induces a scalar product on $L_{[r]}$, which corresponds to the ℓ_2-norm on $L_{[r]}$ with respect to the standard basis.

Each simplex Δ possesses a vector bundle $(T\Delta)_{[r]}$ defined fiberwise as $(T_p\Delta)_{[r]}$, $p \in \Delta$. A *V-valued form* ω (or a *form* when V is understood) on Δ is a choice of a linear function $\omega_p : (T_p\Delta)_{[r]} \to V$ for each $p \in \Delta$ so that for any smooth section s of the bundle $(T\Delta)_{[r]}$, $\omega_p(s(p))$ is a smooth function of p. Now a V-form on X is defined as in Definition 2.3. Generalizing the standard definition of de Rham complex, we let $\Omega^*(X, V)$ be the complex of (smooth) forms on simplicial complex X with the usual differential.

The norm on $(T\Delta)_{[r]}$ gives rise to the sup-norm $|\cdot|_p$ on the forms $\omega_p : (T_p\Delta)_{[r]} \to V$ by the rule $|\omega| := \sup_{|f| \le 1,\ p \in X} |\omega_p(f)|$.

Fix an orthonormal coordinate system $x_1, ..., x_k$ on each simplex Δ^k in X' and equip X with the pull-back coordinate system.

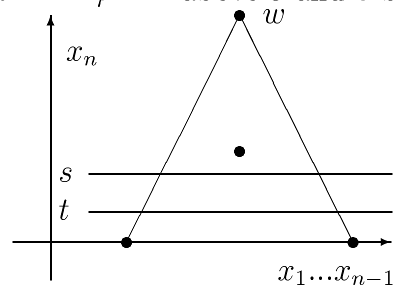
Definition 2.4. *A form ω on X is called* bounded *if $|\omega|$ is bounded by a universal constant.*

Now $\Omega^*_\flat(X, V)$ is the space of bounded forms in $\Omega^*(X, V)$ whose differentials are also bounded. Obviously, $\Omega^*_\flat(X, V)$ is a subcomplex of de Rham complex with the induced differential d. $H^*_\flat(X, V)$ will stand for the homology of $\Omega^*_\flat(X, V)$.

There is a close relationship between forms and simplicial cochains. The well known de Rham's theorem establishes an isomorphism between $H^*(M, \mathbb{R})$ and $H^*_{dR}(M, \mathbb{R})$, the latter being the homology of de Rham complex for a manifold M. In [8], Griffiths and Morgan proved an analogous result for polynomial forms with rational coefficients on simplicial complexes. First, our goal is to show an isomorphism $H^*_\flat(X, V) \cong H^*_{(\infty)}(X, V)$ of ℓ_∞-cohomology (see the definition below) in the case when X is the universal cover of a finite simplicial complex X'. The proof will be a refinement of the argument in [8] with some care taken about boundedness and smoothness.

Definition 2.5. *A* normed *abelian group A is an abelian group with a norm $|\cdot| : A \to \mathbb{R}_+$ satisfying the following properties: (1) $|-a| = |a|$, (2) $|a| = 0 \Leftrightarrow a = 0$, and (3) $|a + b| \le |a| + |b|$.*

FIGURE 1. φ is 1 above s and 0 below t.



Given a normed abelian group A and a complex X as above, a cellular cochain $c \in C^r(X, A)$ is called *bounded* if there is a constant $K = K(c) \geq 0$ so that $|c(\sigma)| \leq K$ for any r-cell σ in X. $C^r_{(\infty)}(X, A)$ will denote the subgroup of all bounded cochains in $C^r(X, A)$. It is an easy exercise to show that $\delta(C^r_{(\infty)}(X, A)) \subseteq C^{r+1}_{(\infty)}(X, A)$, where δ is the coboundary homomorphism in $C^*(X, A)$ (here we use the fact that X has finitely many types of simplices). So $C^*_{(\infty)}(X, A)$ is a chain complex and we denote its homology by $H^*_{(\infty)}(X, A)$.

Let X be the universal cover of a finite n-dinensional simplicial complex X' with the induced simplicial structure, and let G be the fundamental group of X'. As we saw before, X is an $\mathbb{E}$-complex. Given a Banach space V, there is a homomorphism $I : \Omega^r(X, V) \to C^r(X, V)$ obtained by integration over simplices. More precisely, for $\omega \in \Omega^*(X, V)$, we define the r-cochain $I(\omega)$ by $I(\omega)(\sigma) := \int_\sigma \omega$ where σ is an r-simplex in X. This integral is well defined since V is complete.

Up to G-equivariance there are only finitely many simplices in X, so their areas are bounded by a universal constant. It follows that $I(\omega)$ is a bounded cochain whenever ω is a bounded form. Thus I restricts to a homomorphism $I : \Omega^*_\flat(X, V) \to C^*_{(\infty)}(X, V)$. Also, Stokes' theorem precisely states that I commutes with differentials, so I induces a homomorphism $I_* : H^*_\flat(X, V) \to H^*_{(\infty)}(X, V)$ in homology, which we call *the integration map*.

In [11], in the case when X is a manifold, Whitney describes a map $W : C^*(X, V) \to \Omega^*(X, V)$ (φ in the notation of [11]). For our purposes we will need the following modified version of this map. Let Δ be an n-dimensional simplex in $\mathbb{E}^n$ whose edges have length 1, where n is the dimension of X. Let w be a vertex of Δ. Moving Δ by an isometry of $\mathbb{E}^n$ we can arrange that the codimension one face of Δ not containing w lies in the standard hyperplane $\mathbb{E}^{n-1} = \{(x_1, ..., x_n) \in \mathbb{E}^n \mid x_n = 0\}$ of $\mathbb{E}^n$ and that the nth coordinate of w is positive (see. Fig. 1).

Let y be the nth coordinate of the center of mass of Δ. Choose constants t and s with $0 < t < s < y$, and let $f : \mathbb{R} \to \mathbb{R}_+$ be a smooth function with $f((-\infty, t]) = 0$, $f([s, \infty)) = 1$. Define $\varphi(x_1, ..., x_n) := f(x_n)$.

Now for each vertex p in X we describe a smooth function $\varphi_p : X \to \mathbb{R}_+$ as follows. Take an arbitrary simplex σ in X. If p is not a vertex in σ, define $\varphi_p|_\sigma := 0$. If p is a vertex in σ, map σ isometrically onto a face in Δ so that p is taken to w. Note that this isometric embedding of a face is unique up to an isometry of $\mathbb{E}^n$ preserving φ. $\varphi_p|_\sigma$ is then defined as the pull-back of φ on Δ.

Obviously φ_p is a well defined smooth function on X since it is compatible with restrictions to faces, and its support is contained in $Star(p)$. (For a simplex σ in X, $Star(\sigma)$ is the union of the open simplices in X whose closure contains σ as a face.) Also, φ was so chosen that the interiors of the supports of functions $\{\varphi_p \mid p \in X^{(0)}\}$ form a locally finite cover of X. Therefore, we obtain a partition of unity $\{\psi_p \mid p \in X^{(0)}\}$ by taking $\psi_p := \dfrac{\varphi_p}{\sum_{q\in X^{(0)}} \varphi_q}$.

Now, for each k-simplex $\sigma =< p_0, ..., p_k >$ in X, define

$$W(\sigma) := \psi_{p_0},\ if\ k = 0$$

$$W(\sigma) := k! \sum_{i=0}^{k} \psi_{p_i}\ d\psi_{p_0} \wedge ... \wedge \widehat{d\psi_{p_i}} \wedge ... \wedge d\psi_{p_k},\ if\ k \geq 1. \tag{1}$$

(Here by abuse of notation σ is identified with the k-cochain taking value 1 on σ and 0 everywhere else.) More precisely, $W(\sigma)$ is a collection of k-forms $W(\sigma)|_\Delta$ defined on simplices Δ^l of X. Each $W(\sigma)|_\Delta$ can be extended to a neighborhood of Δ^l in $\mathbb{E}^l$ by formula (1), since the functions φ_{p_i} extend. Obviously, the collection $\{W(\sigma)|_\Delta \mid \Delta\ is\ a\ simplex\ in\ X\}$ is compatible with restrictions to faces. Thus, $W(\sigma)$ is a k-form on X. Note that the support of $W(\sigma)$ is contained in $Star(\sigma)$. The map $W : C^k(X, V) \to \Omega^k(X, V)$ is defined by $W(\sum_\sigma \alpha_\sigma \sigma) := \sum_\sigma \alpha_\sigma W(\sigma)$, $\alpha_\sigma \in V$. We call W *the Whitney map.*

Any M_K-complex can be given an $\mathbb{E}$-structure so that the identity map induces a diffeomorphism between the two structures on each simplex (proof inductively on skeleta). It means that the Whitney map can be defined for any M_K-complex X.

Lemma 2.6. *$W : C^*(X, V) \to \Omega^*(X, V)$ is a chain map.*

We only need to show that $W\delta = dW$ for the differentials δ and d in $C^*(X, V)$ and $\Omega^*(X, V)$, respectively. The proof is an exercise or refer to [11], p.140.

The image under W of a bounded simplicial chain is a bounded form, since $|W(\sigma)|$ is bounded independently of σ, and the cover

$$\{supp(W(\sigma)) \mid \sigma\ is\ a\ simplex\ in\ X\}$$

is (uniformly) locally finite. Hence W restricts to a map $W : C^*_{(\infty)}(X, V) \to \Omega^*_\flat(X, V)$. Lemma 2.6 implies that W induces a map $W_* : H^*_{(\infty)}(X, V) \to H^*_\flat(X, V)$ in homology.

Lemma 2.7. *The Whitney map $W : C^*(X, V) \to \Omega^*(X, V)$ is a section of the integration map $I : \Omega^*(X, V) \to C^*(X, V)$. In particular, $W : C^*_{(\infty)}(X, V) \to \Omega^*_\flat(X, V)$ is a section of $I : \Omega^*_\flat(X, V) \to C^*_{(\infty)}(X, V)$.*

Proof. We repeat the proof of an analogous lemma in [11]. If σ is a k-simplex in X, then the support of $W(\sigma)$ is contained in $Star(\sigma)$. Hence, for any k-simplex σ' distinct from

σ,

$$[(I \circ W)(\sigma)](\sigma') = [(I(W)(\sigma))](\sigma') = \int_{\sigma'} W(\sigma) = 0.$$

It remains to prove that, for $\sigma' = \sigma$, $[(I \circ W)(\sigma)](\sigma) = 1$. We show this by induction on the dimension of σ.

If $\sigma =< p >$, then $W(< p >)(< p >) = \varphi_p(p) = 1$. If σ is a k-simplex in X, then $\partial\sigma$ can be viewed as a $(k-1)$-cochain. The k-cochain $\delta(\partial\sigma)$ takes value 1 on σ, so the forms $W(\sigma)$ and $W(\delta(\partial\sigma))$ coincide when restricted to σ. Then, using Lemma 2.6, Stokes' theorem, and the induction hypotheses,

$$[(I \circ W)(\sigma)](\sigma) = \int_\sigma W(\sigma) = \int_\sigma W(\delta(\partial\sigma)) = \int_\sigma dW(\partial\sigma) = \int_{\partial\sigma} W(\partial\sigma) = 1.$$

The last equality holds by the induction hypothesis. Lemma 2.7 is proved. □

3. De Rham's theorem for ℓ_∞-cohomology.

For the rest of this section X' will denote a finite simplicial complex, and X will be the universal cover of X'. As we saw before, X and X' are $\mathbb{E}$-complexes. Our goal is to prove

Theorem 3.1 ("Bounded" de Rham). *The maps*

$$I_* : H^*_\flat(X, V) \to H^*_{(\infty)}(X, V) \quad \text{and} \quad W_* : H^*_{(\infty)}(X, V) \to H^*_\flat(X, V)$$

are mutually inverse isomorphisms for any Banach space V.

Our main tools will be the following bounded versions of the extension lemma and the Poincaré lemma (cf [8]).

Lemma 3.2 (Poincaré lemma, a "bounded" version). *There is a universal constant C such that if $C(L)$ is a simplicial subcomplex in X which is the cone over a finite complex L (with the structure of a simplicial complex induced from L) and ω is a closed form in $\Omega^l_\flat(C(L), V)$, then, for some $\psi \in \Omega^{l-1}_\flat(C(L), V)$,*

(a) $\omega = d\psi$, *and*
(b) $|\psi| \le C|\omega|$.

Proof. The proof is essentially the standard proof of the Poincaré lemma for manifolds. Let w be the vertex of the cone. Given a closed l-form ω on $C(L)$, let $C(\sigma)$ be the cone over a k-dimensional simplex σ in L, and restrict ω to $C(\sigma)$. By definition of a form on a simplex it means that $C(\sigma)$ is viewed as embedded into $\mathbb{E}^{k+1}$ and ω is defined on a neighborhood U of $C(\sigma)$. Without loss of generality we can assume that U is star-shaped with respect to the vertex w of $C(\sigma)$, and that w is the origin 0 in $\mathbb{E}^{k+1}$.

We define a map $H : U \times [0,1] \to U$ by $H(x,t) := tx$. Obviously, H is smooth, $H|_{U\times\{1\}} = id_U$ and $H|_{U\times\{0\}} = 0$, so H is a smooth contraction of U. It follows that ω is exact, i.e. $\omega = d\psi$, where ψ is obtained by integration of ω along straight lines from 0 in $\mathbb{E}^{k+1}$. More precisely, ψ is defined by

$$\psi(v_1 \wedge ... \wedge v_{l-1}) = \int_0^1 \omega(\frac{\partial}{\partial t} \wedge (H \circ i_t)_* v_1 \wedge ... \wedge (H \circ i_t)_* v_{l-1}) dt,$$

where $i_t : U \to U \times [0,1]$ is the inclusion on the t level. Note that ψ is smooth. Since the basis elements $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_k}$ are distorted by inclusions i_t only a bounded amount and ω is bounded, this implies that the value of ψ on the basis of $(T_p\Delta)_{[r-1]}$ is bounded by $C_0|\omega|$ for some universal constant C_0. Then $|\psi| \leq C|\omega|$ for some universal C. If σ' is a face in σ then $C(\sigma')$ is a face in $C(\sigma)$. For each point $p \in C(\sigma')$ the definitions of ψ_p with respect to $C(\sigma')$ and with respect to $C(\sigma)$ coincide since the straight segment $[v,p]$ lies in $C(\sigma')$. Hence ψ is a form on $C(L)$ and Lemma 3.2 is proved. □

Let Δ^n be the standard n-simplex.

Lemma 3.3 (Extension lemma, a "bounded" version)**.** *For any integer n there is a constant $C_n > 0$ such that if φ is a form in $\Omega^l_\flat(\partial\Delta^n, \mathbb{R})$, then there is a form $\bar{\varphi}$ in $\Omega^l_\flat(\Delta^n, \mathbb{R})$ such that*

(a) $\bar{\varphi}\,|_{\partial\Delta^n} = \varphi$,
(b) $|\bar{\varphi}| \leq C_n|\varphi|$, *and*
(c) $|d\bar{\varphi}| \leq C_n\ (\ |\varphi| + |d\varphi|\ )$.

Proof. For $l > n-1$ the statement is obvious since φ is identically zero. So we can assume that $l \leq n-1$.

Step 1. First, let σ^{n-1} be the standard simplex, and α be an l-form on σ^{n-1}. Once and for all, fix a smooth function $h : [0,1] \to \mathbb{R}$ such that $h([0,\frac{1}{2}]) = 0$, $h([\frac{3}{4},1]) = 1$ and $h([0,1]) \subseteq [0,1]$. Denote

$$c_0 := max\{|h|_\infty, |h'|_\infty\} \geq 1$$

($c_0 = 10$ will work) and let $pr : \sigma \times [0,1] \to \sigma$ be the obvious projection. Let $(x_1, ..., x_{n-1})$ denote a coordinate system on σ and x_n be the standard coordinate on $[0,1]$. Denote $\beta(x_1, ..., x_n) = h(x_n) \cdot pr^*(\alpha)(x_1, ..., x_n)$. Note that actually $pr^*(\alpha)$ does not depend on x_n. Obviously, $|\beta|_{\sigma\times[0,1]} \leq |\alpha|_\sigma$, since $|h| \leq 1$.

If $\alpha = \sum_I \alpha_I dx^I$, then

$$d\beta = \sum_{j=1}^n \sum_I \frac{\partial(\alpha_I h)}{\partial x^j} dx^j \wedge dx^I =$$

$$= \sum_I (\alpha_I \frac{\partial h}{\partial x^n} dx^n \wedge dx^I + \sum_{j=1}^{n-1} h \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I) = h' \cdot \alpha + h \cdot d\alpha,$$

so we have $|d\beta| = |h' \cdot \alpha + h \cdot d\alpha| \leq c_0(|\alpha| + |d\alpha|)$.

Step 2. Now let σ be a codimension 1 face of Δ, α be a form on σ, and v be the vertex of Δ not contained in σ. Δ is the join of v and σ. If $\pi : \Delta \setminus \{v\} \to \sigma$ is the stereographic projection from the vertex v and $g : \sigma \times [0,1] \to \Delta$ is the map taking $\sigma \times \{0\}$ to v, $\sigma \times \{1\}$

identically to σ, and segments $\{pt\} \times [0,1]$ linearly to segments $[v, pt]$, then the following diagram commutes.

$$\begin{array}{ccc} \sigma \times (0,1] & \xrightarrow{\ g\ } & \Delta \setminus \{v\} \\ & {}_{pr}\searrow \quad \swarrow_{\pi} & \\ & \sigma & \end{array}$$

Given a form α on σ, by Step 1 there is an extension β on $\sigma \times [0,1]$ with $|\beta| \leq |\alpha|$, $|d\beta| \leq c_0(|\alpha| + |d\alpha|)$. The restriction $g|_{\sigma \times [\frac{1}{2},1]}$ is a diffeomorphism onto its image, so we can define a form $\bar{\alpha}$ on $g(\sigma \times [\frac{1}{2}, 1])$ as the push-forward of β via g, and extend it by zero on the rest of Δ. $\bar{\alpha}$ is obviously an extension of α. Since $g|_{\sigma\times[\frac{1}{2},1]}$ shrinks distances no more than twice,

$$|\bar{\alpha}| = |\bar{\alpha}|_{\sigma\times[\frac{1}{2},1]}| \leq 2^{n-1}|\beta| \leq 2^{n-1}|\alpha| \leq c_1|\alpha|$$

and

$$|d\bar{\alpha}| = |d\bar{\alpha}|_{\sigma\times[\frac{1}{2},1]}| \leq 2^{n-1}|d\beta| \leq 2^{n-1}c_0(|\alpha| + |d\alpha|) = c_1(|\alpha| + |d\alpha|),$$

where $c_1 := 2^{n-1}c_0$. Note an important property following from the construction that if α vanishes on a face τ of σ, then $\bar{\alpha}$ vanishes on the join of τ and v.

Step 3. Now, given a form φ on $\partial\Delta^n$, we enumerate the $(n-1)$-simplices of $\partial\Delta^n$ as $\{\sigma_0, ..., \sigma_n\}$ and define $\alpha_0 := \varphi|_{\sigma_0}$. By step 2, there is an extension $\bar{\alpha}$ of α_0 to Δ^n such that

$$|\bar{\alpha_0}| \leq c_1|\alpha_0| \leq c_1|\varphi|$$

and

$$|d\bar{\alpha_0}| \leq c_1(|\alpha_0| + |d\alpha_0|) \leq c_1(|\varphi| + |d\varphi|).$$

The form $\varphi - \bar{\alpha_0}$ vanishes on σ_0 and (assuming that $c_1 \geq 1$)

$$|\varphi - \bar{\alpha_0}| \leq |\varphi| + |\bar{\alpha_0}| \leq 2c_1(|\varphi|),$$
$$|d\varphi - d\bar{\alpha_0}| \leq |d\varphi| + |d\bar{\alpha_0}| \leq$$
$$|d\varphi| + c_1(|\varphi| + |d\varphi|) \leq 2c_1(|\varphi| + |d\varphi|).$$

Denote $\alpha_1 := (\varphi - \bar{\alpha_0})|_{\sigma_1}$, extend α_1 to $\bar{\alpha_1}$ on Δ so that

$$|\bar{\alpha_1}| \leq c_1|\varphi - \alpha_0| \leq 2c_1^2|\varphi|,$$
$$|d\bar{\alpha_1}| \leq c_1(|\alpha_1| + |d\alpha_1|) \leq$$
$$c_1[2c_1|\varphi| + 2c_1(|\varphi| + |d\varphi|)] \leq 2c_1^2(|\varphi| + |d\varphi|).$$

Denote $\alpha_2 := \varphi - \alpha_0 - \alpha_1$. Since α_1 vanishes on $\sigma_0 \cap \sigma_1$, $\bar{\alpha_1}$ must vanish on σ_0, hence α_2 vanishes on $\sigma_0 \cup \sigma_1$. Continuing this process inductively we get that $\varphi - \sum_{i=0}^{n} \bar{\alpha_i}$ is identically zero on $\partial\Delta^n$ and

$$|\bar{\alpha_i}| \leq (2c_1)^{i+1}|\varphi|,$$

$$|d\bar{\alpha}_i| \le (2c_1)^{i+1}(|\varphi| + |d\varphi|),$$

so by elementary algebra

$$|\sum_{i=0}^{n} \bar{\alpha}_i| \le (\sum_{i=0}^{n}(2c_1)^{i+1})|\varphi| \le 2(2c_1)^{n+1}(|\varphi|),$$

$$|d\sum_{i=0}^{n} \bar{\alpha}_i| \le 2(2c_1)^{n+1}(|\varphi| + |d\varphi|).$$

It shows that $\bar{\varphi} := \sum_{i=0}^{n} \bar{\alpha}_i$ is the desired extension of φ, where $C_n := 2(2c_1)^{n+1} = 2^{n^2+n+1}c_0$. This proves the extension lemma. □

Proof of Theorem 3.1. Lemma 2.7 says that W is a section of I, so I is surjective and we have a short exact sequence

$$0 \to \Omega_0^*(X,V) \to \Omega_\flat^*(X,V) \underset{W}{\overset{I}{\rightleftharpoons}} C^*_{(\infty)}(X,V) \to 0,$$

where $\Omega_0^*(X,V)$ is the kernel of I. The conclusion of Theorem 3.1 is equivalent to saying that the homology of complex $\Omega_0^*(X,V)$ is trivial. Unraveling what it means one can see that for the proof of Theorem 3.1 it only remains to show the following lemma.

Lemma 3.4. *Let X be the universal cover of a finite simplicial complex, and let $\varphi \in \Omega_\flat^k(X,V)$ satisfy $d\varphi = 0$ and $\int_{\Delta^k}\varphi = 0$ for all simplices Δ^k in X. Then there exists $\psi \in \Omega_\flat^{k-1}(X,V)$ such that $d\psi = \varphi$ and $\int_{\sigma^{k-1}}\psi = 0$ for all simplices σ^{k-1} in X.*

Before proving Lemma 3.4 we need the following (cf [8], Lemma 8.4):

Lemma 3.5. *There is a constant C_k depending only on k so that for each closed k-simplex Δ in X the following conditions are satisfied:*

(a_k) *Let φ be a closed form in $\Omega_\flat^r(\Delta,V)$ which vanishes on $\partial\Delta$. If $r = k$, assume also that $\int_\Delta \varphi = 0$. Then $\varphi = d\psi$ for some $\psi \in \Omega_\flat^{r-1}(\Delta,V)$ which vanishes on $\partial\Delta$ and so that $|\psi| \le C_k|\varphi|$.*

(b_k) *Let φ be a closed form in $\Omega_\flat^r(\partial\Delta,V)$, $r > 0$. If $r = k-1$, assume also that $\int_{\partial\Delta}\varphi = 0$. Then $\varphi = d\psi$ for some $\psi \in \Omega_\flat^{r-1}(\partial\Delta,V)$ so that $|\psi| \le C_k|\varphi|$.*

Proof of Lemma 3.5. Now we essentially repeat the argument of [8]. Induction on k. Cases (a_0),(b_0) and (b_1) are obvious.

$\boldsymbol{(a_1)}$ In this case Δ is the segment $[0,1]$. The statement is obvious for $r \ge 2$, since φ is identically 0. For $r = 0$, ϕ is a constant function which vanishes on $\partial\Delta$, so $\varphi = 0$ and the claim follows. For $r = 1$, we have $\int_{\Delta^1}\varphi = 0$ and φ vanishes on $\partial\Delta^1$. Take $\psi(t) := \int_{[0,t]}\varphi$. Then $\psi(0) = 0$ and $\psi(1) = \int_{[0,1]}\varphi = 0$ by the assumption, so $|\psi| \le C_1|\varphi|$ for some constant C_1, and $\varphi = d\psi$.

$\boldsymbol{(a_{k-1}) \Rightarrow (b_k)}$ Given $\Delta = \Delta^k$ and a closed form φ in $\Omega_\flat^r(\partial\Delta,V)$, let σ be a codimension 1 face of Δ. Then $\partial\Delta\setminus int(\sigma)$ is a cone over $\partial\sigma$, so by the Poincaré lemma $\varphi|_{\partial\Delta\setminus int(\sigma)} = d\psi$

for some bounded $(r-1)$-form ψ on $\partial\Delta \setminus int(\sigma)$. Extend ψ to a bounded form $\widetilde{\psi}$ on $\partial\Delta$ using the extension lemma. Then $\varphi - d\widetilde{\psi}$ is a closed r-form on $\partial\Delta$ vanishing on $\partial\Delta \setminus int(\sigma)$.

In the case $r = k-1$, by Stokes' theorem,

$$\int_\sigma (\varphi - d\widetilde{\psi}) = \int_{\partial\Delta} (\varphi - d\widetilde{\psi}) = \int_\Delta d(\varphi - d\widetilde{\psi}) = \int_\Delta 0 = 0.$$

So the hypotheses of (a_{k-1}) are satisfied for the form $(\varphi - d\widetilde{\psi})|_\sigma$, hence $(\varphi - d\widetilde{\psi})|_\sigma = d\mu$ for an $(r-1)$-form μ on σ vanishing on $\partial\sigma$. Extend μ by 0 to a form $\widetilde{\mu}$ on $\partial\Delta$, so $\varphi = d(\widetilde{\psi} + \widetilde{\mu})$. The argument involved only applications of the Poincaré lemma and the extension lemma, so $|\psi| \le C_k |\varphi|$, where C_k is calculated from the constants in the lemmas.

$(b_k) \Rightarrow (a_k)$ Let $\Delta = \Delta^k$ and $\varphi \in \Omega^r_\flat(\Delta, V)$ be a closed r-form vanishing on $\partial\Delta$. By the Poincaré lemma, $\varphi = d\psi$ for some $\psi \in \Omega^{r-1}_\flat(\Delta, V)$. We want to find such a form ψ which vanishes on $\partial\Delta$. Since $d\psi|_{\partial\Delta} = \varphi|_{\partial\Delta} = 0$, then the restriction $\psi|_{\partial\Delta}$ is closed.

In the case $r = 1$, $\psi|_{\partial\Delta}$ is a closed 0-form, i.e. it is a constant function $\psi = C$ on $\partial\Delta$. Replace ψ by $\psi - C$, then $\psi|_{\partial\Delta} = 0$ and $\varphi = d\psi$, as needed. So we can assume $r > 1$. In the case $k = r$,

$$\int_{\partial\Delta} \psi = \int_\Delta d\psi = \int_\Delta \varphi = 0,$$

so the hypotheses of (b_k) are satisfied for the restriction $\psi|_{\partial\Delta}$, hence $\psi|_{\partial\Delta} = d\mu$ for some $\mu \in \Omega^{r-2}_\flat(\partial\Delta, V)$ and also $|\mu| \le C_k|\psi|$. By the extension lemma, μ extends to a form $\widetilde{\mu}$ on Δ. We have $\varphi = d(\psi - d\widetilde{\mu})$ and $(\psi - d\widetilde{\mu})|_{\partial\Delta} = 0$, i.e. $\psi - d\widetilde{\mu}$ is the form we need. To satisfy the property $|\psi - d\widetilde{\mu}| \le C_k|\varphi|$ we enlarge C_k using the constants from the Poincaré lemma and the extension lemma. This proves Lemma 3.5. □

Proof of Lemma 3.4. Let φ be a k-form as in the hypotheses and $n = dim X$. Then for dimension reasons φ vanishes on the $(k-1)$-skeleton $X^{(k-1)}$. For each k-simplex Δ in X, Lemma 3.5(a_k) gives a $(k-1)$-form ψ_Δ so that (1) $d\psi_\Delta = \varphi|_\Delta$, (2) $\psi_\Delta|_{\partial\Delta} = 0$, and (3) $|\psi_\sigma| \le C_k|\varphi|_\Delta| \le C_k|\varphi|$, where C_k depends only on k.

For each simplex σ of dimension less than k define φ_σ to be identically 0. Condition (2) above implies that the collection $\{\psi_\Delta \mid \Delta \textit{ is a simplex in } X\}$ gives a form ψ_1 on $X^{(k)}$ and condition (3) says that ψ_1 is bounded. Using the extension lemma $n-k$ times we extend ψ_1 to all of X preserving boundedness.

Condition (1) says that the form $\varphi - d\psi_1$ vanishes on $X^{(k)}$. Also it is closed since φ is. We apply the same argument to the bounded form $\varphi - d\psi_1$, obtain a bounded form ψ_2 so that $\varphi - d\psi_1 - d\psi_2$ vanishes on $X^{(k+1)}$, and so on. In the end we have $\varphi - d\psi_1 - ... - \psi_{n-k+1}$ is identically 0 on $X = X^{(n)}$. So $\varphi = d(\psi_1 + ... + \psi_{n-k+1})$. This finishes the proof of Lemma 3.4. □

Theorem 3.1 is proved. □

4. Metabolicity of negatively curved complexes.

Definition 4.1. *Given $K \le 0$, a simplicial complex X* has curvature K *if*

- *X is an M_K-complex and*

- *the link of each vertex in X viewed as a spherical complex is a $CAT(1)$ space, i.e. the distance between two points in any geodesic triangle of perimeter $< 2\pi$ in the link is at most that between the corresponding points in the comparison triangle in the unit sphere.*

If $K < 0$, X is called negatively curved.

$S_i(X, R)$ will denote the set of singular i-chains with coefficients in a ring R.

Definition 4.2. *Given an M_K-complex X, a singular chain c in $S_i(X, R)$ is called* smooth *if the image of each singular simplex σ of c is contained in a closed simplex Δ of X and σ is an embedding (with respect to the smooth structure on Δ). $C_i^{sm}(X, R)$ denotes the set of smooth i-chains in X with coefficients in R.*

Given a singular i-boundary b in $S_i(X, R)$, a smooth filling *of b is a smooth chain c in $S_{i+1}(X, R)$ such that $\partial c = b$.*

Each smooth singular simplex σ with image in a simplex Δ of X has a Riemannian metric induced from Δ. Hence $Area(\sigma)$ is well defined (as the integral of the volume form over σ). For an arbitrary smooth filling $c = \sum \alpha_i \sigma_i$, its area is defined by $Area(c) := \sum |\alpha_i| \cdot Area(\sigma_i)$.

Definition 4.3. *A group G is called* $\mathbb{R}$-metabolic *if there is a complex of type $K(G, 1)$ with finite 2-skeleton and the universal cover X so that the following equivalent conditions are satisfied:*

1. $H^2_{(\infty)}(X, V) = 0$ *for any normed vector space V over* $\mathbb{R}$.
2. *The inclusion map $i : Z_1(X, \mathbb{R}) \to C_1(X, \mathbb{R})$ admits a bounded retraction $r : C_1(X, \mathbb{R}) \to Z_1(X, \mathbb{R})$. Here $C_1(X, \mathbb{R})$ is equipped with the ℓ_1-norm and $Z_1(X, \mathbb{R})$ with the filling norm.*

See [4], Theorem 13.9 for the proof of equivalence (it was proved for coefficients $\mathbb{Z}$ and a normed abelian group instead of a vector space V, but the same argument works for $\mathbb{R}$).

The main result of this section is

Theorem 4.4. *The universal cover X of a finite negatively curved simplicial complex X′ admits a piecewise smooth combing with bounded areas. In particular, the fundamental group of X′ is $\mathbb{R}$-metabolic.*

"Bounded areas" here means that each triangle formed by an edge e and two elements of the combing has a smooth filling c such that $Area(c)$ is bounded by a constant independent of e. The main part of the proof is to construct fillings for the triangles. A difficulty here is to make sure that the filling is smooth. We use a combinatorial argument for this.

Proof of Theorem 4.4. Let v be a basepoint in X, and let G be the fundamental group of X'. By simultaneous scaling the metric on the simplices of X we can assume that X has curvature -1. As shown in [1], each point p in X can be connected to v by a unique geodesic arc $[p, v]$ and also all the geodesic triangles in X satisfy the $CAT(-1)$ comparison inequality. It follows then that X is contractible (see [1]), so X' is a finite $K(G, 1)$.

Proposition 7.7 in [5] says that $H^*_{(\infty)}(X,V) = H^*_{(\infty)}(X,\bar{V})$ whenever defined, where V is any normed vector space and $\bar{V}$ is its completion. So our goal is to establish $H^2_{(\infty)}(X,V) = 0$ only for a Banach space V.

By a natural abuse of notation, we identify the edges (= 1-simplices) in the triangulation of X with elements of $C_1(X,\mathbb{Z})$. These elements generate $C_1(X,\mathbb{Z})$. Given such an edge e with endpoints $i(e)$ and $t(e)$, there is a canonical (1-dimensional) triangle b_e formed by the geodesics $[v,i(e)]$, e, and $[t(e),v]$ (subdivided so that each edge lies entirely in a closed simplex). Again, we think of b_e as of the cycle in $C_1^{sm}(X,\mathbb{Z})$ assigning value 1 to each singular 1-simplex.

Lemma 4.5. *For each edge e in X, there exists a smooth disk filling D_e of b_e so that $Area(D_e) \leq Length(e)$.*

Assume this lemma for a moment to finish the proof of the theorem.

Let c be a cocycle representing an element in $H^2_{(\infty)}(X,\mathbb{R})$, i.e. $\delta c = 0$. We take $\varphi := W(c) \in \Omega^2_\flat(X,\mathbb{R})$, where W is the Whitney map. Then $d\varphi = dW(c) = W(\delta c) = 0$, so φ is a closed 2-form on X. For each edge e let D_e be a filling guaranteed by Lemma 4.5, and define

$$a(e) := \int_{D_e} \varphi.$$

Extending a by linearity (over $\mathbb{R}$) we obtain a 1-cochain $a \in C^1(X,\mathbb{R})$.

Now, for any 2-simplex Δ in X, connect the vertices of Δ by geodesics to the basepoint, and fill in the obtained geodesic triangles using Lemma 4.5. Let $D_{\partial\Delta}$ denote the sum of those fillings. Then using the fact that φ is closed and Stokes' theorem,

$$(\delta a)(\Delta) = a(\partial\Delta) = \int_{D_{\partial\Delta}} \varphi = \int_\Delta \varphi = I(\varphi)(\Delta),$$

so $\delta a = I(\varphi) = I(W(c)) = c$. In other words, c is a coboundary.

It remains to show that a is bounded. But for any edge e, by Lemma 4.5,

$$a(e) = \int_{D_e} \varphi \ \leq |\varphi| Area(D_e) \leq \ |\varphi| Length(e) \leq$$

$$|\varphi| \cdot max\{Length(e) \mid e \text{ } is \text{ } an \text{ } edge \text{ } in \text{ } X\}$$

and $max\{Length(e) \mid e \text{ } is \text{ } an \text{ } edge \text{ } in \text{ } X\}$ is a universal constant. This proves Theorem 4.4 assuming Lemma 4.5.

Now we go to

Proof of Lemma 4.5. Parameterize e by $t \in [0,1]$. For each t, let $\alpha_t : [0,1] \to [v,e(t)] \subset X$ be the map of constant speed onto the unique geodesic $[v,e(t)]$. Define $\alpha(t,s) := \alpha_t(s)$. The $CAT(-1)$ property implies that α is continuous as a function of two variables.

Let B be a large enough finite subcomplex of X containing the image of α. B is the disjoint union of its open simplices. Fix some t and let σ be an open simplex of B, and let α_t be the corresponding geodesic in B. Consider the preimage $I_\sigma := \alpha_t^{-1}(\sigma) \subset [0,1]$. First, this preimage must be convex in $[0,1]$, since σ is convex in B (this follows from the

$CAT(-1)$ property). So I_σ is either a non-degenerate interval or a point. In the former case I_σ must be open in $[0,1]$. Indeed, suppose that one of its endpoints, say, $p \in (0,1)$ is such that $\alpha(p) \in \sigma$ and the points just away from p (on one side) are mapped to an open simplex having σ as a face. In this case we can shorten α_t near p, which contradicts the assumption that α_t is geodesic.

The discussion above suggests the following definition.

Definition 4.6. *Given a map $\beta : [0,1] \to B$ of constant speed onto a geodesic in B, a* pattern *for β in $[0,1]$ is*

- *a finite sequence $0 = s_0 < s_1 < ... < s_m = 1$ so that the intervals*
$$(s_0, s_1), \{s_1\}, (s_1, s_2), \{s_2\}, ..., \{s_{m-1}\}, (s_{m-1}, s_m)$$
are mapped by β to distinct open simplices of B, and also
- *a prescription is given to which open simplices of B these intervals are mapped by β.*

The points s_i will be called the vertices *of the pattern.*

As we saw, each geodesic α_t gives rise to a pattern which we denote by P_t. For a subset A in X, $carr(A)$ will denote the open simplex containing A (if it exists). We call two patterns, $P_t = \{0 = s_0 < s_1 < ... < s_m = 1\}$ and $P_{t'} = \{0 = s'_0 < s'_1 < ... < s'_m = 1\}$, *equivalent*, $P_t \approx P_{t'}$, if they are preimages of the same sequence of open simplices in B, i.e. $carr(\alpha_t(s_0)) = carr(\alpha_{t'}(s'_0))$, $carr(\alpha_t(s_0, s_1)) = carr(\alpha_{t'}(s'_0, s'_1))$, ..., $carr(\alpha_t(s_m)) = carr(\alpha_{t'}(s'_m))$. We say that a pattern P_t is *a limit* of the pattern $P_{t'}$ if there is a sequence $t_i \to t$ in $[0,1]$ such that $P_{t_i} \approx P_{t'}$.

Now fix a pattern P and define
$$I_P := \{t \in [0,1] \mid P_t \approx P\},$$
i.e. I_P is the "preimage" of P in $[0,1]$. Let $x := \inf I_P$, $y := \sup I_P$. If $x = y$, draw the geodesic $[e(x), v]$ and go to another pattern P. Suppose now $x \neq y$. Patterns P_x and P_y are limits of P, i.e. there are sequences $x_i \to x$ and $y_i \to y$ with $P_{x_i} \approx P \approx P_{y_i}$.

For each i, the pattern P_{x_i} is induced by α_{x_i} on the vertical segment $\{x_i\} \times [0,1]$ (see Fig.2). The vertices of the pattern are s^i_j. Since $[x,y] \times [0,1]$ is compact, by taking a subsequence of x_i several times we can assume that s^i_j converges to a point s_j on $\{x\} \times [0,1]$ when $i \to \infty$. Those limit points s_j give almost a pattern on $\{x\} \times [0,1]$ with the only exception that some of the points s_j may coincide. Still, the order of these points along $\{x\} \times [0,1]$ is preserved. The same argument applied to the sequence $\{y_i\}$ gives a pattern on $\{y\} \times [0,1]$ with points s'_j on it. Now we note that both $\alpha_x(s_j)$ and $\alpha_y(s'_j)$ lie in the closure of the open simplex $carr(\alpha_{x_i}(s_j)) = carr(\alpha_{y_i}(s'_j))$, so we can connect $\alpha_x(s_j)$ and $\alpha_y(s'_j)$ by a geodesic segment in this closure, for each j. Also, each geodesic quadrilateral $[\alpha_x(s_j), \alpha_y(s'_j)]$, $[\alpha_y(s'_j), \alpha_y(s'_{j-1})]$, $[\alpha_y(s'_{j-1}), \alpha_x(s_{j-1})]$, $[\alpha_x(s_{j-1}), \alpha_x(s_j)]$ lies in a closed simplex of B. If the vertices of this quadrilateral are distinct, draw the diagonal $[\alpha_x(s_j), \alpha_y(s'_{j-1})]$ (see Fig. 3). Each "elementary" geodesic triangle obtained in this way may be filled in by taking the convex hull of its vertices in the closed simplex of B containing it. We call $D_{[x,y]}$ the filling obtained by this procedure. Each of the 2-simplices

FIGURE 2. In the domain of α.

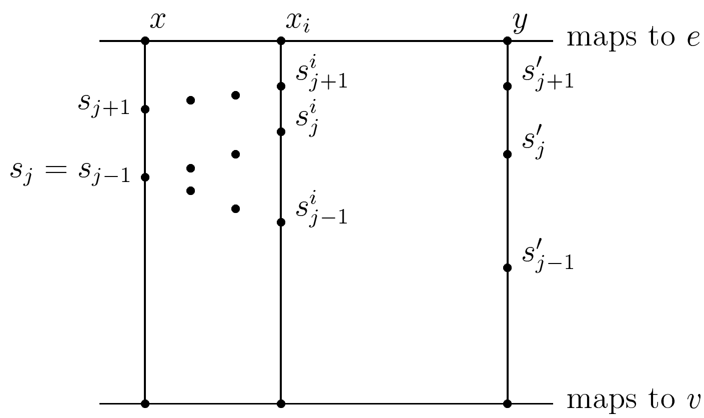
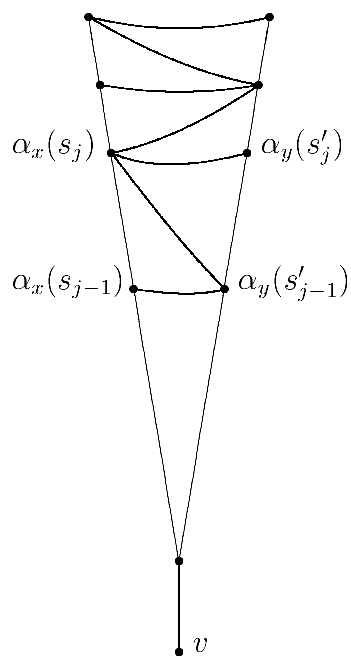


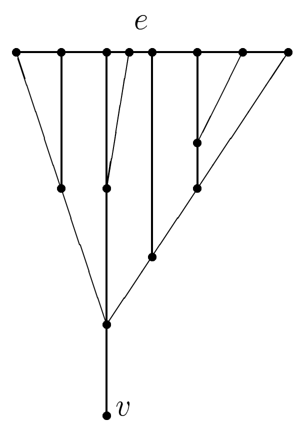
FIGURE 3. The filling of a simple triangle.



in $D_{[x,y]}$ possesses a metric of curvature -1 induced from the metric on the simplices of B, i.e. the 2-simplices are isometric to triangles in $\mathbb{H}^2$.

Now pattern P is not equivalent to those in $\{P_t \mid t \in [0,1] \setminus [x,y]\}$, so we can apply the procedure described above to the triangles $b_{[e(0),e(x)]}$ and $b_{[e(y),e(1)]}$, and so on. This process will terminate since there are only finitely many equivalence classes of patterns P_t. As the result, the triangle b_e is represented as a finite "concatenation" of "narrow" triangles (see Fig. 4). These triangles may partially or completely degenerate, but after removing

FIGURE 4. Concatenation of simple triangles.



the degenerate parts we can assume that each of the "narrow" triangles is *simple* in the following sense:

Definition 4.7. *Let γ be a geodesic segment in a finite pointed complex B of curvature -1. The geodesic triangle b_γ is called* simple *if it admits a smooth filling D_γ with the following properties:*

- *Viewed as an abstract simplicial 2-complex, D_γ is a triangulated 2-disk each of whose 2-simplices is mapped homeomorphically onto the convex hull of three points in a closed simplex of B.*
- *The vertices of D_γ lie on the two geodesic sides connecting the endpoints of γ to the basepoint.*

It suffices only to prove Lemma 4.5 for simple triangles:

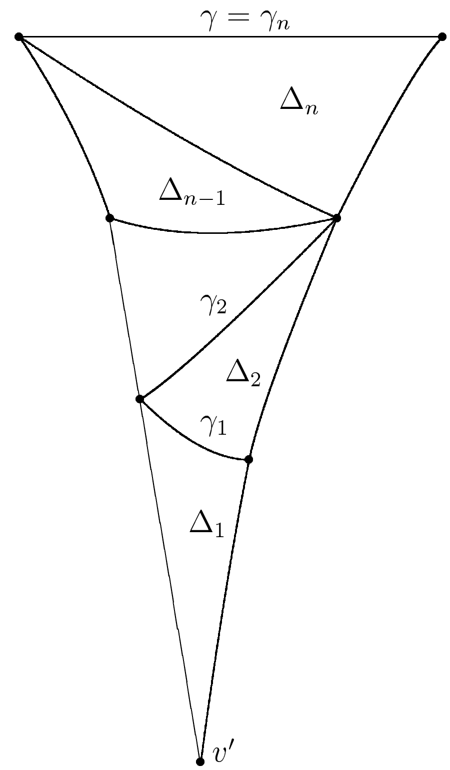
Lemma 4.8. *For any simple triangle b_γ, $Area(D_\gamma) \leq Length(\gamma)$, where D_γ is the filling of b_γ constructed above.*

Proof of Lemma 4.8. In the filling D_γ enumerate the 1- and 2-simplices starting from the vertex v' nearest to the basepoint as shown in Fig. 5. We see that the angles at the vertices of D_γ lying in the interiors of the geodesic sides are at least π, since otherwise we would be able to shorten the sides.

Let D_{γ_i} be, as before, the filling of the geodesic triangle of b_γ (i.e. D_{γ_i} is just the union of 2-simplices $\Delta_1, ..., \Delta_i$). Start with an isometric embedding $f_1 : \Delta_1 \to \mathbb{H}^2$. There is a unique way to extend it to an isometric embedding $f_2 : \Delta_1 \cup \Delta_2 \to \mathbb{H}^2$. Next, there is a unique extension $f_3 : \Delta_1 \cup \Delta_2 \cup \Delta_3 \to \mathbb{H}^2$ which is an isometric embedding when restricted to $\Delta_2 \cup \Delta_3$. Iterating this procedure we obtain a map $f : D_\gamma \to \mathbb{H}^2$ which is an isometric embedding when restricted to each union $\Delta_i \cup \Delta_{i+1}$. Let Γ_i denote the convex hull of $f(\gamma_i)$ and $f(v')$ in $\mathbb{H}^2$.

By induction on i we will show that f is injective on D_{γ_i} and that $f(D_{\gamma_i}) \subseteq \Gamma_i$.

FIGURE 5.



For $i = 1$ the statement is obvious. Assume the statement for i. Let l be the infinite geodesic containing vertices w and w' as shown on Fig. 6. Consider $D_{\gamma_{i+1}} = D_{\gamma_i} \cup \Delta_{i+1}$. Since the angle θ between the sides $[w', w]$ and $[w', w'']$ in $f(D_{\gamma_{i+1}})$ is at least π and the triangles $f(\Delta_i)$ and $f(\Delta_{i+1})$ lie on the opposite sides of the geodesic $f(\gamma_i)$, we see that the segment $[w', w'']$ must be in the shaded corner. It follows that $f|_{D_{\gamma_{i+1}}}$ is injective and $f(D_{\gamma_{i+1}}) \subseteq \Gamma_{i+1}$.

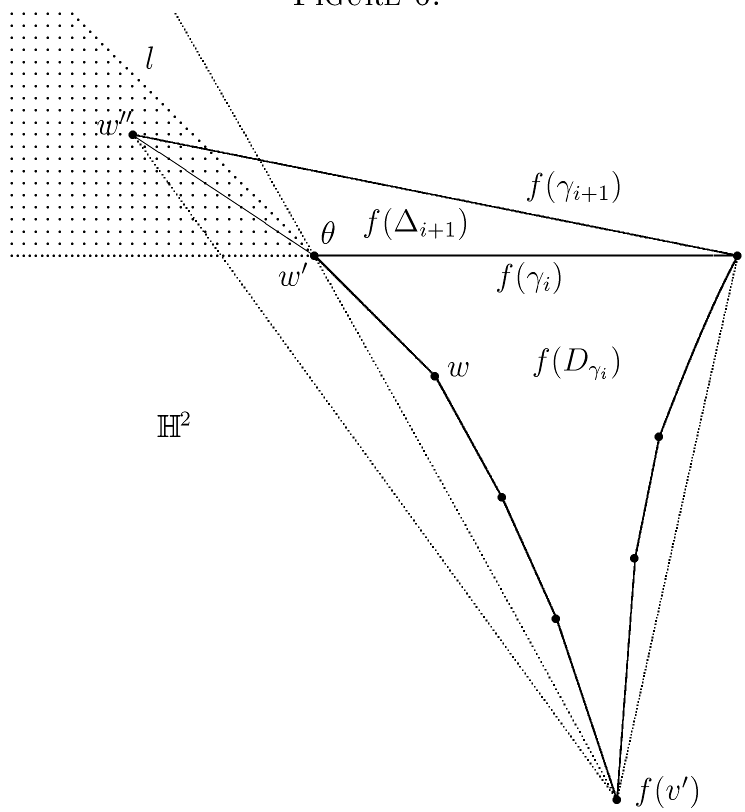
Hence in the end, for $i = n$, we get

$$Area(D_\gamma) \leq Area(f(D_\gamma)) \leq Area(\Gamma_n).$$

In $\mathbb{H}^2$, the area of a geodesic triangle is bounded by the length of any side (see [5, Lemma 12.4]). So the filling D_γ satisfies $Area(D_\gamma) \leq Area(\Gamma_n) \leq Length(\gamma)$, and Lemma 4.8 follows. □

This finishes the proofs of Lemma 4.5 and Theorem 4.4. □

FIGURE 6.



5. HIGHER DIMENSIONAL ISOPERIMETRIC FUNCTIONS.

Definition 5.1. *Let X be a cell complex.* The ℓ_1-norm $|\cdot|_1$ *on $C_i(X,\mathbb{R})$, is defined by $|\sum_\sigma \alpha_\sigma \sigma|_1 := \sum_\sigma |\alpha_\sigma|$, and* the filling norm $|\cdot|_f$ *on $B_i(X,\mathbb{R})$ is defined by*

$$|b|_f := inf\{|a|_1 \mid a \in C_{i+1}(X,\mathbb{R}) \text{ and } \partial a = b\}$$

for $b \in B_i(X,\mathbb{R})$.

As shown in [3], in the case when X is the universal cover of a $K(G,1)$ with finite $(i+1)$-skeleton,

$$|\cdot|_f \geq A|\cdot|_1 \tag{2}$$

on $B_i(X,\mathbb{R})$ for some universal constant $A > 0$. In particular, $|\cdot|_f$ is a norm.

Definition 5.2. *Let X be a contractible cell complex. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is called* a homological isoperimetric function for i-cycles *or* isoperimetric function*, for shortness, if $|\cdot|_f \leq g(|\cdot|_1)$ on $Z_i(X,\mathbb{R})$.*

Remark. The property of having linear isoperimetric function in dimension i can be described homologically [6, Theorem 6.1], and hence is a quasiisometry invariant. Therefore it makes sense to talk about a group having linear isoperimetric function.

Definition 5.3. *Given a cell complex X with a basepoint vertex v,* an $\mathbb{R}$-combing on X *is an assignment of a chain $q_w \in C_1(X,\mathbb{R})$ to each vertex w of X so that $\partial q_w = w - v$. An $\mathbb{R}$-combing is called* quasigeodesic *if there exist $S > 0$ and a quasigeodesic combing $\{p_w \mid w \in X^{(0)}\}$ on $X^{(1)}$ so that supp(q_w) lies in the S-neighborhood of supp(p_w).*

We say that a group G admits an $\mathbb{R}$-combing with bounded areas *if there exist a cell complex of type $K(G,1)$ with universal cover X, $T > 0$, and an $\mathbb{R}$-combing $\{q_w\}$ on X such that, for any edge e in X, $|q_{i(e)} + e - q_{t(e)}|_f \leq T$. ($i(e)$ and $t(e)$ here are the initial and terminal vertices of e, respectively.)*

The main result of this section is

Theorem 5.4. *Let G be a finitely presented group. If G admits a quasigeodesic $\mathbb{R}$-combing with bounded areas, then each inclusion map $B_i(X,\mathbb{R}) \to C_i(X,\mathbb{R})$, $i \geq 1$, admits a bounded retraction. In particular, G has a linear isoperimetric function for real cycles in each positive dimension.*

Here B_i is equipped with the filling norm and C_i with the ℓ_1-norm. The following two theorems will be deduced as corollaries of the main result.

Theorem 5.5. *Let G be the fundamental group of a finite negatively curved complex. Then G has a linear isoperimetric function for real cycles in each positive dimension.*

Theorem 5.6 ([7])**.** *Let G be the fundamental group of a closed manifold of negative sectional curvature. Then G has a linear isoperimetric function for real cycles in each positive dimension.*

Theorem 5.6 was proved in [7] by geometric methods. Note that the cases of a complex group and a manifold group are not implied by each other, since we consider manifolds of non-constant negative curvature.

Given a cell complex X, we will always assign length 1 to each of the edges and put the path metric on the 1-skeleton. For $x \in X^{(1)}$, $B_x(r)$ will denote the *cellular ball* of radius r centered at x, i.e. it is the union of all cells in X whose vertices lie within distance r from x. The r-neighborhood of any set in $X^{(1)}$ is defined analogously. Let C_i be *the maximal size of* $(\leq i)$*-cells*, i.e. C_i is the maximum of the distances between vertices of a cell, maximum taken over all cells of dimension $\leq i$.

Following [2] we call a group *combable* if it admits a quasigeodesic combing with the fellow-traveler property.

Lemma 5.7 (Projection lemma.)**.** *Let G be a combable group and X be the universal cover of a $K(G,1)$ with finitely many cells in each dimension. Then there exists a sequence of affine functions $R_i : \mathbb{R}_+ \to \mathbb{R}_+$ and constants L_i such that for any $r > 0$ and $x \in X^{(1)}$ there exists a chain map*

$$pr_* = pr_{x,r*} : C_*(X,\mathbb{R}) \to C_*(B_x(R_i(r)),\mathbb{R})$$

so that (1) $|pr_*(a)|_1 \leq L_i|a|_1$ for any $a \in C_i(X,\mathbb{R})$, and (2) the restriction of pr_* to $C_*(B_x(r),\mathbb{R})$ is identity.

Remark. A $K(G,1)$ with finitely many cells in each dimension always exists for a combable group as shown in [2].

Proof of Projection lemma. Obviously, x can be taken to be a vertex. Let $\{p_w\}$ be a (λ,ϵ)-quasigeodesic combing on $X^{(1)}$ based at the vertex x. We can assume that p_w maps into $X^{(0)}$. Also, we extend p_w to a map $p_w : [0,\infty) \to X^{(0)}$ so that the restriction $p_w|_{[0,t_w]}$ is a (λ,ϵ)-quasigeodesic emanating from x and $p_w([t_w,\infty)) = w$ for some t_w.

Fix $r > 0$ and $x \in X^{(1)}$. We will extend the argument used in [9] to construct a cellular map $pr_{x,r} = pr : X \to X$ inductively on skeleta.

For $w \in X^{(0)}$ define $pr(w) := p_w(\lambda r + \epsilon)$. It is easy to see that, for $w \in B_x(r)^{(0)}$, $pr(w) = w$, and also, for any $w' \in X^{(0)}$, $pr(w') \in B_x(\lambda(\lambda r+\epsilon)+\epsilon)$. Define $R_0(r) := \lambda(\lambda r+\epsilon)+\epsilon$, $L_0 := 1$.

Now for each edge e we choose a shortest edge path γ connecting $pr(i(e))$ to $pr(t(e))$, and map e to γ. This extends pr to the 1-skeleton. The fellow-traveler property gives an upper bound δ_1 on the number of edges of γ. Subdivide e into $\leq \delta_1$ edges so that pr maps 1-cells to 1-cells of X. Define $R_1(r) := R_0(r) + \delta_1$, $L_1 := \delta_1$.

We proceed inductively on dimension, mapping each i-cell Δ to a filling of $pr(\partial\Delta)$. Subdivide Δ so that i-cells of Δ are mapped to cells (possibly of smaller dimension). There are uniform bounds δ_i on the size of $pr(\Delta)$ and L_i on the number of i-cells in the subdivision of Δ. Define $R_i(r) := R_{i-1}(r) + \delta_i$. Now pr is defined for $X^{(i)}$ and $pr(X^{(i)}) \subseteq B_x(R_i(r))$, hence pr induces a map $pr_* : C_i(X,\mathbb{R}) \to C_i(B_x(R_i(r)),\mathbb{R})$ with the required properties. Since G acts on $X^{(i)}$ cocompactly, R_i and L_i can be chosen independently of x. Projection lemma is proved. ☐

Proof of Theorem 5.4. "Bounded areas" condition implies that G satisfies the linear isoperimetric inequality for 1-cycles over $\mathbb{R}$. It follows from the argument in [3] that G is hyperbolic in this case. In particular, G is combable.

Let X be the universal cover of a $K(G,1)$ with finitely many cells in each dimension. It is an easy exercise to show that having a quasigeodesic $\mathbb{R}$-combing with bounded areas is a quasiisometry invariant property, hence X admits such an $\mathbb{R}$-combing $\{q_w\}$. Pick a basepoint vertex v.

Lemma 5.8. *Let X be as above. Then for any $r > 0$ and an integer $i \geq 1$ there exist constants $R = R_i(r) > 0$ and $M = M_i(r) > 0$ such that for any ball $B_x(r) \subseteq X$ and for any i-cycle z supported in $B_x(r)$ there exists a filling a of z supported in $B_x(R)$ so that $|a|_1 \leq M|z|_1$.*

Proof. We let $R_i(r)$ be the function from Projection lemma. Pick $B := B_x(r) \subseteq X$. For $z \in Z_i(B,\mathbb{R})$ define

$$|z|_{f,B} := \inf\{|a|_1 \mid a \in C_{i+1}(B_x(R_i(r)),\mathbb{R}) \text{ and } \partial a = z\}.$$

X is contractible, hence each cycle z in B may be filled with a chain a in X. Projection lemma implies that a can be projected into the ball $B_x(R_i(r))$ so that $\partial a = z$ is preserved.

Therefore z always has a filling supported in $B_x(R_i(r))$, i.e. $|\cdot|_{f,B} < \infty$. Also $|\cdot|_{f,B} \geq |\cdot|_f$, by definition, hence $|\cdot|_{f,B}$ is a norm.

The norms $|\cdot|_{f,B}$ and $|\cdot|_1$ are equivalent on $Z_i(B,\mathbb{R})$ as two norms on a finite dimensional vector space, hence there exists a constant M, depending on i,r, and x, such that $|z|_{f,B} \leq M|z|_1$ for any $z \in Z_i(B,\mathbb{R})$. For a fixed r, the set $\{B_x(r) \subseteq X \mid x \in X^{(1)}\}$ is finite up to equivariance, hence M can be chosen independently of x. Since $C_{i+1}(B_x(R(r)),\mathbb{R})$ is finite dimensional, the norm $|z|_{f,B}$ is realized by a filling a of z, i.e. $|a|_1 = |z|_{f,B} \leq M|z|_1$. Lemma 5.8 is proved. □

Lemma 5.9. *Let X be as above. Then there is a retraction $j : C_*(X,\mathbb{R}) \to Z_*(X,\mathbb{R})$ for the inclusion $Z_*(X,\mathbb{R}) \subseteq C_*(X,\mathbb{R})$ with the following property: for each $i \geq 1$ there exist constants $S_i > 0$ and $T_i > 0$ such that for each i-cell Δ there is a filling a_Δ of $j(\Delta)$ so that supp(a_Δ) lies in the S_i-neighborhood of any geodesic connecting a vertex of Δ to the basepoint v, and $|a_\Delta|_1 \leq T_i$.*

Proof. Recall that C_i is the maximal size of $(\leq i)$-cells in X.

We construct j inductively on i.

$\underline{i = 1}$ The $\mathbb{R}$-combing $\{q_w\}$ is S-close to a quasigeodesic combing $\{p_w\}$. Since quasi-geodesics are close to geodesics in hyperbolic spaces, we can assume (changing S if needed) that $\{p_w\}$ is geodesic. We also make S large enough so that any two geodesics in $X^{(1)}$ with corresponding endpoints at distance ≤ 1 are uniformly S-close to each other.

For an edge e we define $j(e) := q_{i(e)} + e - q_{t(e)}$ and extend j by linearity on $C_1(X,\mathbb{R})$. It is easy to see that $j(e)$ is a 1-cycle, and that j is identity on the 1-cycles, i.e. j is indeed a retraction of the inclusion $Z_1(X,\mathbb{R}) \subseteq C_1(X,\mathbb{R})$. Since the $\mathbb{R}$-combing is with bounded areas, $|j(e)|_f \leq T$ for some T independent of e. Then there exists a filling a of $j(e)$ with $|a|_1 \leq T + 1$. The only problem is that *supp*(a) may not lie in a neighborhood of $p_{i(e)}$. We fix this as follows.

Let c_1 be the restriction of a to $B_v(2C_2)$, i.e. c_1 takes the same values as a in $B_v(2C_2)$ and 0 everywhere else. Similarly, let h_1 be the restriction of $\partial a = j(e)$ to $B_v(2C_2)$. Denote $b_1 := \partial c_1 - h_1$. Then *supp*$(b_1) \subseteq B_v(2C_2)$. Next we define c_k inductively as the restriction of $a - c_1 - ... - c_{k-1}$ to $B_v(2C_2k)$ and h_k as the restriction of $\partial a - h_1 - ... - h_{k-1}$ to $B_v(2C_2k)$. Denote $Slice_k := B_v(2C_2k) \backslash B_v(2C_2(k-1))$. Now *supp*$(\partial c_k - h_k) \subseteq Slice_k \cup Slice_{k-1}$. Since $Slice_k$ and $Slice_{k-1}$ are disjoint, $\partial c_k - h_k$ decomposes as $\partial c_k - h_k = b_k + b'_k$, where b_k and b'_k are supported in $Slice_k$ and $Slice_{k-1}$, respectively (see Fig 7). We continue this until $k = m$, m being the least integer so that $e \subseteq B_v(2C_2m)$. Define $c_m := a - c_1 - ... - c_{m-1}$, $h_m := \partial a - h_1 - ... - h_{m-1}$, and $Slice_m := X \setminus B_v(2C_2(m-1))$. Since the $Slice$'s are disjoint and

$$\sum_k (b_{k-1} - b'_k) = \sum_k (\partial c_{k-1} - h_k) = \partial \sum_k c_k - \sum_k h_k = \partial a - \partial a = 0,$$

we must have $b'_k = b_{k-1}$. Here we assume $b_0 = b_m = 0$ by definition.

Summarizing the discussion above, a has a decomposition $a = \sum_k c_k$ with $supp(c_k) \subseteq Slice_k$, and also $\partial c_k = h_k + b_k - b_{k-1}$, where $supp(b_k) \subseteq Slice_k$. Pick a geodesic p in $X^{(1)}$ connecting v to a vertex of e. Each $supp(h_k) \subseteq supp(\partial a)$ lies in the S-neighborhood of p.

Let $x_k := p(2C_2k - C_2)$ and $r := 3C_2 + S$. Then $supp(h_k) \subseteq B_{x_k}(r)$. By Projection lemma each of c_k, h_k, b_k is mapped by $pr_{x_k,r*}$ to $B_{x_k}(R_2(r))$. Let $\bar{c}_k := pr_{x_k,r*}(c_k)$, $\bar{b}_k := pr_{x_k,r*}(b_k)$, $\tilde{b}_k := pr_{x_k,r*}(b_{k-1})$. By (1) in the Projection lemma,

$$|\bar{c}_k|_1 \leq L_2|c_k|_1 \tag{3}$$

and by (2), $pr_{x_k,r*}$ fixes h_k and ∂h_k. Also $\partial h_k = \partial(\partial c_k - b_k + b_{k-1}) = -\partial b_k + \partial b_{k-1}$ and $supp(\partial b_k)$ and $supp(\partial b_{k-1})$ are disjoint, hence both supports lie in $supp(\partial h_k) \subseteq B_{x_i}(r)$, so $pr_{x_k,r*}$ fixes ∂b_k as well.

$$\partial(\bar{b}_k - \tilde{b}_{k+1}) = pr_{x_k,r*}(\partial b_k) - pr_{x_{k+1},r*}(\partial b_k) = \partial b_k - \partial b_k = 0,$$

so $\bar{b}_k - \tilde{b}_{k+1}$ is a 1-cycle, and also

$$supp(\bar{b}_k - \tilde{b}_{k+1}) \subseteq B_{x_k}(r) \cup B_{x_{k+1}}(r) \subseteq B_{x_k}(5C_2 + S),$$

then by Lemma 5.8 there is a filling d_k of $\bar{b}_k - \tilde{b}_{k+1}$ with $supp(d_k) \subseteq B_{x_k}(R)$, $R := R_1(5C_2 + S)$ and $|d_k|_1 \leq M|\bar{b}_k - \tilde{b}_{k+1}|_1$, $M := M_1(5C_2 + S)$. Then using 2

$$\begin{aligned} |d_k|_1 &\leq M|\bar{b}_k - \tilde{b}_{k+1}|_1 \leq M(|\bar{b}_k|_1 + |\tilde{b}_{k+1}|_1) \leq \\ &ML_2(|b_k|_1 + |b_k|_1) \leq ML_2 2A|c_k|_1. \end{aligned} \tag{4}$$

For $a_e := \sum_k(\bar{c}_k + d_k)$,

$$\begin{aligned} \partial a_e = \sum_k(\partial pr_{x_k,r*}(c_k) + \partial d_k) = \sum_k(pr_{x_k,r*}(\partial c_k) + \bar{b}_k - \tilde{b}_{k+1}) = \\ \sum_k(pr_{x_k,r*}(\partial c_k) + pr_{x_k,r*}(b_k)) - \sum_k pr_{x_{k+1},r*}(b_k) = \\ \sum_k(pr_{x_k,r*}(\partial c_k) + pr_{x_k,r*}(b_k) - pr_{x_k,r*}(b_{k-1}) = \\ \sum_k pr_{x_k,r*}(h_k) = \sum_k h_k = \partial a, \end{aligned}$$

so a_e is a filling of $\partial a = j(e)$, and $supp(a_e)$ lies in the S_1-neighborhood of the geodesic p, where $S_1 := \max\{R_2(3C_2 + S), R_1(5C_2 + S)\}$. Hence the same is true for any geodesic from v to an endpoint of e (changing S_1 if needed). By (3) and (4)

$$|a_e|_1 \leq \sum_k(|\bar{c}_k|_1 + |d_k|_1) \leq \sum_k(L_2 + ML_2 2A)|c_k|_1 = (L_2 + ML_2 2A)|z|_1.$$

Take $T_1 := L_2 + ML_2 2A$. Then a_e is the required filling of $j(e)$.

<u>$i \Rightarrow i+1$</u> We assume that $j : C_i(X, \mathbb{R}) \to Z_i(X, \mathbb{R})$ is constructed and satisfies the induction hypotheses with constants S_i and T_i. Let $C := C_{i+1}$ be the size of $(\leq i+1)$-cells, and S be the fellow-traveler constant for geodesics in $X^{(1)}$. The following constants

will be used in the proof ($R_i(r)$ and $M_i(r)$ are the functions from Lemma 5.8):

$$S' := S_i + S, \quad r := 3S' + 2C, \quad r' := r + S' + C, \quad R' := R_i(3S' + C),$$
$$R'' := R_{i+1}(r + R'), \quad M' := max\{M_i(r), M_{i+1}(R'')\}$$

Take any $(i+1)$-cell Δ and suppose $\partial\Delta = \sum_l \beta_l \sigma_l,\ \beta_l \in \mathbb{R}$. By induction hypotheses each $j(\sigma_l)$ has a filling a_{σ_l} and also j is the identity on i-cycles, hence

$$\partial(\sum_l \beta_l a_{\sigma_l}) = \sum_l \beta_l \partial a_{\sigma_l} = \sum_l \beta_l j(\sigma_l) = j(\sum_l \beta_l \sigma_l) = j(\partial\Delta) = \partial\Delta, \tag{5}$$

i.e. $\sum_l \beta_l a_{\sigma_l}$ is a filling of $\partial\Delta$. Also S' was so chosen that the support of this filling lies in the S'-neighborhood of p. By (5) the chain $z := \Delta - \sum_l \alpha_l a_{\sigma_l}$ is an $(i+1)$-cycle. We define $j(\Delta)$ to be z and extend j by linearity to $C_{i+1}(X, \mathbb{R})$.

Let p be a geodesic edge path from v to a vertex of Δ. Denote $x_k := p(2rk - r)$, $y_k := p(2rk)$, $k = 1, 2, ...$ The balls $B_{x_k}(r')$ cover $supp(z)$ (see Fig. 7). Let c_1 be the restriction of z to $B_{x_1}(r')$, i.e. it is the $(i+1)$-chain taking the same values as z on the cells in $B_{x_k}(r')$ and identically 0 on the rest of cells in X. Since z was a cycle, an exercise on the triangle inequality gives

$$supp(\partial c_1) \subseteq [B_{x_1}(r') \setminus B_{x_1}(r' - C)] \cap B_p(S') \subseteq B_{y_1}(3S' + 2C) = B_{y_1}(r),$$

where $B_p(S')$ denotes the S'-neighborhood of p. Analogously, let c_2 be the restriction of $z - c_1$ to $B_{x_2}(r')$, and now we have

$$\begin{aligned} supp(\partial c_2) \subseteq\ & [B_{x_2}(r') \setminus B_{x_2}(r' - C)] \cap B_p(S') \subseteq \\ & B_{y_1}(3S' + 2C) \cup B_{y_2}(3S' + 2C) = B_{y_1}(r) \cup B_{y_2}(r). \end{aligned}$$

We continue the same way until $z - c_1 - c_2 - ... - c_m = 0$, so z has a decomposition $z = c_1 + c_2 + ... + c_m$ such that $supp(c_k) \subseteq B_{x_k}(r')$ and $supp(\partial c_k) \subseteq B_{y_{k-1}}(r) \cup B_{y_k}(r)$. By our choice of the points y_k the balls $B_{y_{k-1}}(r)$ and $B_{y_k}(r)$ are disjoint, hence ∂c_k decomposes as $\partial c_k = b_k + b'_k$ with $supp(b_k) \subseteq B_{y_k(r)}$ and $supp(b'_k) \subseteq B_{y_{k-1}}(r)$. By renumbering

$$\sum_k (b'_k + b_{k-1}) = \sum_k (b'_k + b_k) = \sum_k \partial c_k = \partial z = 0,$$

and the terms in the first sum are supported in disjoint sets $B_{y_k}(r)$, then $b_{k-1} = -b'_k$. Each b_k is an i-cycle with support in $B_{y_k}(r)$, so by Lemma 5.8 there is a filling a_k of b_k with $supp(a_k) \subseteq B_{y_k}(R')$ such that $|a_k|_1 \leq M'|b_k|_1$.

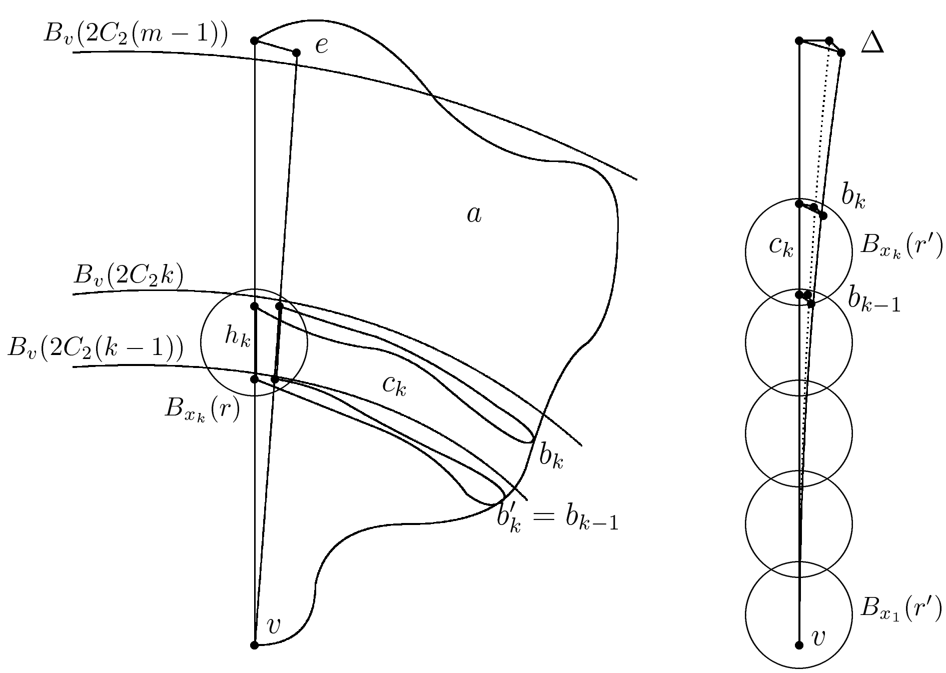
The chain $z_k := c_k - a_k + a_{k-1}$ is an $(i+1)$- cycle since $\partial(c_k - a_k + a_{k-1}) = \partial c_k - b_k + b_{k-1} = \partial c_k - b_k - b'_k = 0$, and also

$$supp(z_k) \subseteq B_{x_k}(r') \cup B_{y_k}(R') \cup B_{y_{k-1}}(R') \subseteq B_{x_k}(r + R'),$$

so again by Lemma 5.8 we can fill z_k by an $(i+2)$-chain d_k with $supp(d_k) \subseteq B_{x_k}(R'')$ and $|d_k|_1 \leq M'|z_k|_1$.

The chain $a_\Delta := \sum_k d_k$ is a filling of z since $\partial a_\Delta = \sum_k \partial d_k = \sum_k z_k = \sum_k c_k = z$, and $supp(a_\Delta) \subseteq \bigcup_k B_{x_k}(R'') \subseteq B_p(R'')$ implies

FIGURE 7. The base of induction $i = 1$ and the induction step $i \Rightarrow i+1$ pictured for $i = 1$.



$$supp(a_\Delta) \subseteq B_{p'}(R'' + S) \tag{6}$$

for any geodesic p' connecting v to a vertex of Δ. Also

$$\begin{aligned} |a_\Delta|_1 &\leq \sum_k |d_k|_1 \leq M'(\sum_k |c_k|_1 + 2\sum_k |a_k|_1) \leq \\ & M'(\sum_k |c_k|_1 + 2M' \sum_k |b_k|_1) \leq M'(\sum_k |c_k|_1 + 2M' \sum_k |\partial c_k|_1) \leq \\ & M'(\sum_k |c_k|_1 + 2M'A \sum_k |c_k|_1) = M'(1+2M'A) \sum_k |c_k|_1 = \\ & M'(1+2M'A)|z|_1 \leq M'(1+2M'A) n_{i+1} T_i, \end{aligned} \tag{7}$$

where n_{i+1} is the maximal number of codimension 1 faces for $(i+1)$-sells Δ in X. Formulas (6) and (7) say that a_Δ satisfies the required properties for $S_{i+1} := R'' + S$ and $T_{i+1} := M'(1+2M'A)n_{i+1}T_i$. Lemma 5.9 is proved. □

The last conclusion of Lemma 5.9 says that j gives a bounded retraction to the inclusion $B_i(X, \mathbb{R}) = Z_i(X, \mathbb{R}) \subseteq C_i(X, \mathbb{R})$. Also this lemma implies that, for any i-cycle $z =$

$\sum_\Delta \beta_\Delta \Delta$ in X,

$$|z|_f \ = |j(z)|_f \leq \sum_\Delta |\beta_\Delta j(\Delta)|_f \leq \sum_\Delta |j(\Delta)|_f \cdot |\beta_\Delta| \leq$$
$$\sum_\Delta |a_\Delta|_1 \cdot |\beta_\Delta| \leq T_i \sum_\Delta |\beta_\Delta| \leq T_i |z|_1,$$

so Theorem 5.4 follows. □

Proof of Theorem 5.5. As before, if X' is the negatively curved finite complex with fundamental group G, we can take X to be $\widetilde{X'}$. Again, by scaling the metric we can assume curvature $K = -1$.

By Theorem 5.4 we only need to exhibit a quasigeodesic $\mathbb{R}$-combing with bounded areas.

In section 2 we defined the Whitney map $W : C^i(X, \mathbb{R}) \to \Omega^i(X, \mathbb{R})$ for any M_K-complex X. A proper way to think about W is as of a map which spreads i-forms from $X^{(i)}$ to X. Also, the supports of these forms are not spread too far if the simplices of X are universally bounded. Now we define "the adjoint map" $W' : C_i^{sm}(X, \mathbb{R}) \to C_i(X, \mathbb{R})$ from smooth chains to simplicial chains as follows: for $s \in C_i^{sm}(X, \mathbb{R})$ and $c \in C^i(X, \mathbb{R})$, $W'(s)$ is determined by $< c, W'(s) > := < W(c), s >$. Unraveling what this nonsense means one can see that the simplicial chain $W'(c)$ is defined by (taking $c = \sigma$, i.e. by abuse of notation c is the cochain having value 1 on σ and 0 everywhere else)

$$W'(s)(\sigma) := < W(\sigma), s > = \int_s W(\sigma). \tag{8}$$

In dimension 0 this just means a discrete sum. One should think of W' as of a map projecting smooth i-chains from X to $X^{(i)}$.

Let C be the maximal diameter of simplices in X.

Lemma 5.10. *Let X be a M_K-complex. Then*

(a) $W' : C_*^{sm}(X, \mathbb{R}) \to C_*(X, \mathbb{R})$ *is a chain map,*
(b) *supp*$(W'(s))$ *lies in the C-neighborhood of supp*(s),
(c) *there exists $N > 0$ such that $|W'(s)|_1 \leq N \cdot Area(s)$ for any smooth singular simplex s, and*
(d) *for any simplex σ in X (thought of both as a smooth and as a simplicial chain),* $W'(\sigma) = \sigma$.

Proof. (a) Since W is a chain map, by Lemma 2.6 and Stokes' theorem we have the following tautology:

$$< c, W'(\partial s) > = < W(c), \partial s > = < dW(c), s > =$$
$$< W(\delta c), s > = < \delta c, W'(s) > = < c, \partial W'(s) >$$

for any cochain c, hence $W'(\partial s) = \partial W'(s)$, W' is a chain map.

It suffices to show (b) only in the case when s is a smooth simplex contained in a simplex Δ of X. Take $c = \sigma$ as before. By the definition of W the support of form $W(\sigma)$ is contained in $Star(\sigma)$. If *supp*(s) does not lie entirely in the C-neighborhood of

$supp(\sigma)$ then $supp(s) \cap Star(\sigma) = \emptyset$. Formula (8) then implies that $W'(s)(\sigma) = 0$. Thus $supp(W'(s))$ is in the C-neighborhood of $supp(s)$.

(c) follows from formula (8), where N is the universal bound for the norm of the form $W(\sigma)$.

(d) Recall that by Lemma 2.7 W is a section of the integration map I. Then we have (thinking of σ as of a cochain)

$$W'(\sigma')(\sigma) = \int_{\sigma'} W(\sigma) =< I(W(\sigma)), \sigma' >=< \sigma, \sigma' > .$$

Then $W'(\sigma') = \sigma'$. □

For a vertex w, p_w will denote the geodesic in X connecting the basepoint v to w. Then p_w is a smooth 1-chain. Let $q_w := W'(p_w)$. By Lemma 5.10(d),

$$\partial q_w = \partial(W'(p_w)) = W'(\partial p_w) = W'(w - v) = W'(w) - W'(v) = w - v.$$

This says that $\{q_w\}$ is an $\mathbb{R}$-combing on $X^{(1)}$. It is quasigeodesic from Lemma 5.10(b) and from the fact that $\{p_w\}$ lies close to a quasigeodesic combing on $X^{(1)}$ (because the inclusion $X^{(1)} \hookrightarrow X$ is a quasiisometry).

As before, for any edge e in X we form a canonical geodesic triangle b_e as the concatenation of $p_{i(e)}$,e, and $p_{t(e)}$. Orient b_e consistently with e. By Lemma 4.5, b_e admits a smooth filling $D_e \in C_2^{sm}(X, \mathbb{R})$ of area bounded by C, so using Lemma 5.10

$$|q_{i(e)} + e - q_{t(e)}|_f = |W'(p_{i(e)}) + W'(e) - W'(p_{t(e)})|_f = |W'(b_e)|_f =$$
$$|W'(\partial D_e)|_f = |\partial W'(D_e)|_f \leq |W'(D_e)|_1 \leq N \cdot Area(D_e) \leq NC,$$

so the combing $\{q_w\}$ is with bounded areas. Theorem 5.5 is proved. □

Proof of Theorem 5.6. Let M be a closed smooth manifold of negative curvature and X be the universal cover of M. By scaling the metric we can assume curvature $K \leq -1$. The space X is homeomorphic to $\mathbb{H}^n$, in particular, X is contractible. X admits a canonical geodesic combing with bounded areas (see [5, Lemma 12.4]), where "geodesic" and "areas" are understood in the sense of the Riemannian metric on X. Also, M admits a smooth triangulation [10, Theorem 10.6], so we can view X as a simplicial complex by lifting the triangulation of M to X. Now we want to apply the argument we used for complexes in the proof of Theorem 5.5. The only obstacle to this is that X may not be a M_K-complex with respect to its triangulation, so the Whitney map W is not defined yet.

We modify the definition of W as follows. $\{Star(w) \mid w \in M^{(0)}\}$ is a finite open cover of M and each $Star(w)$ lifts to M. Pick a smooth partition of unity on M subordinate to this open cover. It lifts to a partition of unity $\{\psi_w \mid w \in X^{(0)}\}$ on X. Define W the same way we did in section 2 (1) with respect to this partition of unity. Construct a quasigeodesic $\mathbb{R}$-combing with bounded areas following the proof of Theorem 5.5 word by word. Theorem 5.6 is proved. □

Actually, the argument above says more. Suppose the hypotheses of Theorem 5.4 are satisfied. Then there exists a bounded retraction $j : C_i(X, \mathbb{R}) \to B_i(X, \mathbb{R})$ for $i \geq 1$, and a diagram chasing argument implies that $H^{i+1}_{(\infty)}(X, V) = 0$ for any normed vector space V over $\mathbb{R}$. So we have the following strengthening of theorems 5.5 and 5.6.

Theorem 5.11. *Let G be the fundamental group of either*

(1) *a finite negatively curved complex, or*
(2) *a closed manifold of negative sectional curvature.*

Then $H^n_{(\infty)}(G, V) = 0$ for any $n \geq 2$ and any normed vector space V over $\mathbb{R}$.

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