BOUNDED COHOMOLOGY CHARACTERIZES HYPERBOLIC GROUPS

by IGOR MINEYEV†

(*University of South Alabama, Department of Mathematics & Statistics, ILB* 325, *Mobile, AL* 36688, *USA*)

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Abstract

A finitely presentable group G is hyperbolic if and only if the map $H_b^2(G, V) \to H^2(G, V)$ is surjective for any bounded G-module. The 'only if' direction is known and here we prove the 'if' direction. We also consider several ways to define a linear homological isoperimetric inequality.

1. Introduction

The question of cohomological description of hyperbolicity was considered by S. M. Gersten who proved the following theorem.

THEOREM 1 ([**3**]). *The finitely presented group G is hyperbolic if and only if $H^2_{(\infty)}(G, \ell_\infty) = 0$.*

Here $H^n_{(\infty)}(G, V)$ is the ℓ_∞-*cohomology* defined by bounded (not necessarily equivariant) cellular cochains in the universal cover of a $K(G, 1)$ complex with finitely many cells in the dimensions up to n. This theorem was generalized by the author [**10**] to higher dimensions: if G is hyperbolic then $H^n_{(\infty)}(G, V) = 0$ for any $n \geqslant 2$ and any normed vector space V (over $\mathbb{Q}$ or $\mathbb{R}$).

The *bounded cohomology* of a group is defined by bounded equivariant cochains in the homogeneous bar construction (see the definition in the next section). B. E. Johnson [**8**, Theorem 2.5] characterized *amenable* groups by the vanishing of $H^1(L^1(G), X^*)$, the first cohomology of the Banach algebra $L^1(G)$. (The vanishing in higher dimensions also follows from his argument. Bounded cohomology is an example of the cohomology above.) In [**11**, p. 1068] G. A. Noskov also characterized amenable groups by the vanishing of the bounded cohomology for the positive dimensions. We present this result in the following form.

THEOREM 2 (Johnson [**8**]). *For a group G the following statements are equivalent.*

(a) *G is amenable.*

(b) $H_b^1(G, V^*) = 0$ *for any bounded G-module V.*

(c) $H_b^i(G, V^*) = 0$ *for any $i \geqslant 1$ and any bounded G-module V.*

† E-mail: mineyev@math.usouthal.edu; http://www.math.usouthal.edu/∼mineyev/math/

The expression 'bounded G-module V' here is what we call a 'bounded Banach $\mathbb{R}G$-module', and V^* is the space conjugate to V.

The main result of this paper is in a sense an analogue of the above two theorems: we characterize hyperbolic groups by bounded cohomology. It was shown in [**9**] that if G is a hyperbolic group then the map $H_b^i(G, V) \to H^i(G, V)$, induced by inclusion, is surjective for any bounded $\mathbb{Q}G$-module V and any $i \geqslant 2$ (see the definitions in the next section). In the present paper we show the converse. When G is finitely presentable, the surjectivity of the above maps only in dimension 2 implies hyperbolicity. Namely, we prove the following.

THEOREM 3 *For a finitely presentable group G, the following statements are equivalent.*

(a) *G is hyperbolic.*

(b) *The map $H_b^2(G, V) \to H^2(G, V)$ is surjective for any bounded G-module V.*

(c) *The map $H_b^i(G, V) \to H^i(G, V)$ is surjective for any $i \geqslant 2$ and any bounded G-module V.*

Here by a 'bounded G-module' we mean any one of the ten concepts $\mathcal{M}_1(G)$ to $\mathcal{M}_{10}(G)$ described in section 5, see Theorem 9 for a more precise statement.

It is quite interesting that the same property can be characterized by the ℓ_∞-cohomology and the bounded cohomology, two theories which do not seem to have much in common. Also, the characterizations of hyperbolic and amenable groups seem strikingly similar. This similarity may be worth further investigation.

There are two crucial points in our proof. First, for finitely presentable groups hyperbolicity is equivalent to the existence of a linear isoperimetric inequality *for real* 1*-cycles*. This equivalence was proved by Gersten. (It follows, for example, from [**3**, Proposition 3.6, Theorem 5.1, and Theorem 5.7] (stated above).) This isoperimetric inequality for real cycles is a homological version of the usual combinatorial isoperimetric inequality *for loops*. We give a direct proof of the fact that the linear isoperimetric inequality for filling (usual) real 1-cycles with *summable* 2-chains implies hyperbolicity. Secondly, one needs to pick appropriate coefficients V. We take V to be the space of all boundaries of summable 2-chains in a cell complex with a nice G-action. Similar coefficients (but $\mathbb{Z}$-modules rather than $\mathbb{R}$-modules or $\mathcal{C}$-modules), and the universal cocycle associated with them were used by S. M. Gersten [**7**, Chapter 12] to introduce *$\mathbb{Z}$-metabolic* (or simply *metabolic*) groups.

2. Preliminaries

Let $\mathbb{F}$ stand for one of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{A}$ stand for one of the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

2.1. *Abelian group norms*

A *normed abelian group A* is an abelian group with an *abelian group norm* $|\cdot| : A \to \mathbb{R}_+$ satisfying

- $|a| = 0$ if and only if $a = 0$, and
- $|a + a'| \leqslant |a| + |a'|$

for all $a, a' \in A$.

2.2. *Norms*

A *normed vector space* W over $\mathbb{F}$ is a vector space with a *norm* $|\cdot| : W \to \mathbb{R}_+$ satisfying

- $|w| = 0$ if and only if $w = 0$,
- $|w + w'| \leqslant |w| + |w'|$, and
- $|\alpha w| = |\alpha| \cdot |w|$

for all $w, w' \in W$ and $\alpha \in \mathbb{F}$.

Of course, each norm on W is an abelian group norm, but not conversely.

2.3. ℓ_1*-norm*

Let a free $\mathbb{A}$-module M have a preferred basis $\{m_i, i \in I\}$. The ℓ_1*-norm* $|\cdot|_1$ on M (with respect to this basis) is given by

$$\left|\sum_{i\in I} \alpha_i m_i\right|_1 := \sum_{i\in I} |\alpha_i|.$$

The ℓ_1-norm is an abelian group norm. Moreover, if M happens to be a vector space over $\mathbb{F}$, then it is a norm.

2.4. ℓ_∞*-norm*

Suppose that $(W, |\cdot|_1)$ and $(W', |\cdot|)$ are normed vector spaces, where W is equipped with the ℓ_1-norm $|\cdot|_1$ with respect to some preferred basis $\{w_i,\ i \in I\}$. For a linear map $\varphi : W \to W'$, the ℓ_∞*-norm* of φ, $|\varphi|_\infty$, is the operator norm of φ, that is, $|\varphi|_\infty$ is the smallest number K (possibly infinity) such that $|\varphi(w)| \leqslant K|w|_1$ for each $w \in W$. One checks that

$$|\varphi|_\infty = \sup_{i\in I} \bigl|\varphi(w_i)\bigr|.$$

2.5. *Cell complexes*

By a *cell complex* we mean a combinatorial cell complex, that is, the one in which the boundary of each cell σ is cellulated and the gluing map of σ restricts to homeomorphisms on the open cells of this cellulation. In particular, each 2-cell can be viewed as a polygon.

We always put the path metric, which is induced by assigning length 1 to each edge, on the 1-skeleta of cell complexes. A cellular ball of radius r centred at a vertex x is the union of all closed cells whose vertices lie r-close to x.

If X is a cell complex and $C_i(X, \mathbb{A})$ is the space of cellular i-chains, we always give $C_i(X, \mathbb{A})$ the ℓ_1-norm with respect to the standard basis consisting of i-cells.

2.6. *Filling norm for compactly supported chains*

Gersten introduced this useful concept (sometimes also called ‘standard norm’).

If G is a finitely presentable group, then there exists a contractible cell complex X with a free cellular G-action such that the induced action on the 2-skeleton of X is cocompact. In particular, the boundary homomorphism $\partial_2 : C_2(X, \mathbb{A}) \to C_1(X, \mathbb{A})$ is bounded (with respect to the ℓ_1-norms in the domain and the target).

The *filling norm* $|\cdot|_{fc,\mathbb{A}}$ on the space $B_1(X,\mathbb{A})$ of 1-boundaries is given by

$$|b|_{fc,\mathbb{A}} := \inf\bigl\{\, |a|_1 \,\bigm|\, a \in C_2(X,\mathbb{A}) \text{ and } \partial a = b\bigr\}.$$

In other words, $|\cdot|_{fc,\mathbb{A}}$ is the abelian group norm induced by the map $\partial_2 : C_2(X,\mathbb{A}) \to C_1(X,\mathbb{A})$ on its image. One checks that $|\cdot|_{fc,\mathbb{Z}}$ is an abelian group norm, and $|\cdot|_{fc,\mathbb{Q}}$ and $|\cdot|_{fc,\mathbb{R}}$ are norms. (The condition $|w| = 0$ if and only if $w = 0$ holds because $|\cdot|_{fc,\mathbb{A}}$ dominates the ℓ_1-norm on $B_1(X,\mathbb{A})$.)

2.7. $\boldsymbol{\ell_1}$-*completion*

For X as above and $i = 1, 2$, let $C_i^{(1)}(X,\mathbb{F})$ be the set of all (absolutely) summable i-chains in X. Since the G-action on the 2-skeleton of X is cocompact, the boundary homomorphism $\hat{\partial}_i : C_i^{(1)}(X,\mathbb{F}) \to C_{i-1}^{(1)}(X,\mathbb{F})$, $i = 1, 2$, is well defined and bounded with respect to the ℓ_1-norm in the domain and in the target. Also, $\hat{\partial}_i$ commutes with the G-action.

Write $B_i^{(1)}(X,\mathbb{F}) := \operatorname{Im}\hat{\partial}_{i+1}$, $Z_i^{(1)}(X,\mathbb{F}) := \operatorname{Ker}\hat{\partial}_i$.

2.8. *Filling norm for summable chains*

Now we consider 'the complete version' of the filling norm. The norm $|\cdot|_{f1,\mathbb{F}}$ on the space $B_1^{(1)}(X,\mathbb{F})$ is given by

$$|b|_{f1,\mathbb{F}} := \inf\bigl\{|a|_1 \,\bigm|\, a \in C_2^{(1)}(X,\mathbb{F}) \text{ and } \partial a = b\bigr\}.$$

Again, since $\hat{\partial_2} : C_2^{(1)}(X,\mathbb{F}) \to C_1^{(1)}(X,\mathbb{F})$ is bounded, $|\cdot|_{f1,\mathbb{F}}$ dominates $|\cdot|_1$ on $B_1^{(1)}(X,\mathbb{F})$ and therefore $|\cdot|_{f1,\mathbb{F}}$ is indeed a norm.

2.9. *Bounded cohomology*

An $\mathbb{F}G$-module is called *bounded* if it is normed as a vector space over $\mathbb{F}$ and G acts on it by linear operators of uniformly bounded norms. For a bounded $\mathbb{F}G$-module V, the *bounded cohomology* of G with coefficients in V, $H_b^*(G,V)$, is the homology of the cochain complex

$$0 \longrightarrow C_b^0(G,V) \xrightarrow{\delta_0} C_b^1(G,V) \xrightarrow{\delta_1} C_b^2(G,V) \xrightarrow{\delta_2} \dots,$$

where

$$C_b^i(G,V) := \{\alpha : G^{i+1} \to V \mid \alpha \text{ is a bounded } G\text{-map}\} \tag{1}$$

and the coboundary map δ_i is defined by

$$\delta_i\alpha\bigl([x_0,\dots,x_{i+1}]\bigr) := \sum_{k=0}^{i+1}(-1)^k\alpha\bigl([x_0,\dots,\widehat{x_k},\dots,x_{i+1}]\bigr). \tag{2}$$

Here G^{i+1} is considered with the diagonal G-action by left multiplication, and by *bounded G-map* we mean a G-map whose image is bounded with respect to the norm on V.

Equivalently, $C_b^i(G,V) \subseteq \operatorname{Hom}_{\mathbb{F}G}(C_i(G,\mathbb{F}),V)$ is the subspace of all $\mathbb{F}G$-morphisms $C_i(G,\mathbb{F}) \to V$ which are bounded as linear maps, where $C_i(G,\mathbb{F})$ is the space of all chains (that is, finite support functions) $G^{i+1} \to \mathbb{F}$ given the ℓ_1-norm with respect to the standard basis G^{i+1}. The coboundary homomorphism δ in $C_b^i(G,V)$ is the dual of the boundary morphism ∂ in $C_i(G,\mathbb{F})$. The spaces $C_b^i(G,V)$ are naturally given the ℓ_∞-norm dual to the ℓ_1-norm on $C_i(G,\mathbb{F})$.

3. Summable 1-chains

In this section we prove that certain summable 1-chains on a graph can be approximated in a good way by good compactly supported chains. These results (though not in full generality) will be needed in section 4 for isoperimetric inequalities. Essentially, we generalize the following theorem.

THEOREM 4 (Allcock–Gersten [**1**]). *If Γ is a graph and f is a summable real-valued 1-cycle on Γ, then there is a countable coherent family C of simple circuits in Γ and a function $g : C \to [0, \infty)$ such that $f = \sum_C g(c)c$.*

'Coherent' here means that, for any such f, $|f|_1 = \sum_C g(c)|c|_1$.

Let Γ be a graph. As a part of the structure, we assign an orientation to each edge of Γ. Therefore 'a 1-chain on Γ' is the same thing as 'a function on the edges of Γ'. The orientation on Γ determines *the initial vertex ιe* and *the terminal vertex τe* for each edge e. A *directed path p* in Γ is a sequence of edges $(e_1, \dots, e_n)$ with $\tau e_i = \iota e_{i+1}$. The initial vertex ιp of the path p is ιe_1 and the terminal vertex τp of p is τe_n. A directed path $(e_1, \dots, e_n)$ is *simple* if the vertices $\iota e_1, \dots, \iota e_n$ are all distinct. If T is a set of vertices in Γ, we say that p is a *T-path* if it is a directed path in Γ such that $\iota p, \tau p \in T$ or $\iota p = \tau p$ (in other words, the homological boundary of p is supported in T).

Let f be a(n absolutely) summable 1-chain in Γ. For a vertex v in Γ, define

$$\mathrm{Div}_v(f) := \sum_{\iota e = v} f(e) - \sum_{\tau e = v} f(e),$$

where e stands for edges in Γ (with the fixed orientation).

By $\Gamma(f)$ we denote the same graph Γ but with an orientation on the edges chosen so that $f(e) \geqslant 0$ for each edge e. Let $\Gamma_+(f)$ be the minimal subgraph of $\Gamma(f)$ containing all the edges e with $f(e) \neq 0$ (that is, $f(e) > 0$).

LEMMA 5 *Let Γ be a graph and T be a set of vertices in Γ. If h is a summable real 1-chain such that* $\mathrm{supp}(\partial h) \subseteq T$ *and $\Gamma_+(h)$ contains no non-trivial simple T-paths, then $h = 0$.*

Proof. (cf. [**1**, Lemma 3.2]). Suppose to the contrary that $h \neq 0$, that is, there is an edge e' in $\Gamma_+(h)$ (hence $h(e') > 0$). Let Γ' be the (non-empty) union of all directed paths $(e_1, \dots, e_n)$ in $\Gamma_+(h)$ with $e_1 = e'$. Then e' is the only edge in Γ' incident on $\iota e'$ (otherwise there would be a non-trivial directed edge cycle in $\Gamma' \subseteq \Gamma_+(h)$).

Case 1. We assume for the moment that Γ' does not contain vertices from T, except possibly for $\iota e'$. Let h' be the summable chain in Γ which coincides with h on Γ' and takes the value 0 outside. If v is any vertex in Γ' different from $\iota e'$, then $v \notin T \supseteq \mathrm{supp}(\partial f)$ and also, by definition, Γ' contains *all* the edges e of $\Gamma_+(h)$ with $\iota e = v$, so

$$0 = \mathrm{Div}_v(h) \leqslant \mathrm{Div}_v(h').$$

Obviously, for the vertex $\iota e'$ the strong inequality holds:

$$0 < h(e') = h'(e') = \mathrm{Div}_{\iota e'}(h'). \tag{3}$$

Since h' is (absolutely) summable, rearranging the terms we get

$$\begin{aligned} 0 \leqslant \sum_{v \text{ vertex in } \Gamma} \mathrm{Div}_v(h') &= \sum_{v \text{ vertex in } \Gamma} \left(\sum_{\iota e = v} h'(e) - \sum_{\tau e = v} h'(e) \right) \\ = \sum_{e \text{ edge in } \Gamma} \big(h'(e) - h'(e)\big) &= 0. \end{aligned}$$

Thus $\mathrm{Div}_v(h') = 0$ for *any* v, which contradicts (3).

Case 2. If Γ' contains s vertex v' from T and $v' \neq \iota e'$, then do the same construction in the other direction: let Γ'' be the union of all the paths $(e_1, \dots, e_n)$ in $\Gamma_+(h)$ with $e_n = e'$. Now Γ'' does not contain a vertex from T, except possibly for $\tau e'$, because otherwise we could connect such a vertex to v' with a T-path in $\Gamma_+(h)$, which would contradict the assumptions of the lemma. So the same argument as in case 1 works for Γ''.

Each T-path p in $\Gamma(f)$ gives rise to an integer 1-chain in Γ which we also denote by p. In the following theorem, $\mathbb{A}$ denotes $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

THEOREM 6 *Let Γ be a graph, T a set of vertices in Γ and f a summable 1-chain on Γ with coefficients in $\mathbb{A}$ and* $\mathrm{supp}(\partial f) \subseteq T$.

(a) *There is a countable family $P = \{p_1, p_2, \dots\}$ of simple T-paths in $\Gamma_+(f)$ and a sequence $\{\alpha_i\}$ in $[0, \infty) \cap \mathbb{A}$ such that $f = \sum_i \alpha_i p_i$ and $|f|_1 = \sum_i \alpha_i |p_i|_1$.*

(b) *If the above f happens to have finite support, then P can be chosen to be finite.*

This is a generalization of Theorem 4 and our proof of Theorem 6 follows the lines of [**1**, proof of Theorem 3.3] (cf. also [**4**, Lemma 3.6]), though we make the proof a bit more explicit without referring to the Zorn's lemma.

Proof. Since f is summable, $\Gamma_+(f)$ has only countably many edges. Let $P = \{p_1, p_2, \dots\}$ be the (countable) set of all simple T-paths in $\Gamma_+(f)$. If the support of f is finite, then P is finite.

For two summable 1-chains f' and f'' in Γ we will write $f' \preceq f''$ if $f'(e) \leqslant f''(e)$ for each edge in $\Gamma(f)$ (with the orientation defined in $\Gamma(f)$).

Let α_1 be the (non-negative) real number which is maximal among those satisfying

$$\alpha_1 p_1 \preceq f.$$

Note that $\alpha_1 \in [0, \infty) \cap \mathbb{A}$ since α_1 coincides with the minimal value of f on the edges of p_1. Continue inductively: if $\alpha_1, \dots, \alpha_{i-1}$ are constructed, let α_i be the maximal real number satisfying

$$\alpha_1 p_1 + \cdots + \alpha_{i-1} p_{i-1} + \alpha_i p_i \preceq f.$$

Inductively we see that each α_i is in $[0, \infty) \cap \mathbb{A}$.

Write $f_i := \alpha_1 p_1 + \cdots + \alpha_i p_i$. Since all the chains are non-negative,

$$|f_i|_1 = \alpha_1 |p_1|_1 + \cdots + \alpha_i |p_i|_1.$$

The sequence f_i is monotone and bounded by the summable chain f, so the chain

$$\bar{f} := \sum_i \alpha_i p_i$$

is well defined and satisfies

$$|\bar{f}|_1 := \sum_i \alpha_i |p_i|_1.$$

Also, $0 \preceq \bar{f} \preceq f$, therefore $0 \preceq f - \bar{f} \preceq f$.

We prove now that $\Gamma_+(f - \bar{f})$ does not contain non-trivial simple T-paths. If it contained a non-trivial simple T-path p, then $\alpha p \preceq f - \bar{f} \preceq f$ for some $\alpha > 0$. Therefore p would be a simple path in $\Gamma_+(f)$ as well, that is, $p = p_i$ for some i, and

$$f_i + \alpha p_i \preceq \bar{f} + \alpha p_i \preceq f.$$

This contradicts the maximality in the definition of α_i.

Also $\operatorname{supp}(\partial(f - \bar{f})) \subseteq \operatorname{supp}(\partial f) \cup \operatorname{supp}(\partial \bar{f}) \subseteq T$ so, by Lemma 5, $f - \bar{f} = 0$, that is, $f = \bar{f} = \sum_i \alpha_i p_i$ and $|f|_1 = |\bar{f}|_1 = \sum_i \alpha_i |p_i|_1$. This finishes the proof of Theorem 6.

4. Linear isoperimetric inequalities

One could mean many different things by 'a linear homological isoperimetric inequality'. In this section we attempt to present a comprehensive list of possible interpretations and show that they are all equivalent to hyperbolicity. Most of this was shown by Gersten in various papers [**1, 3, 5, 7**]; we just collect the statements into one theorem and sketch the proofs.

THEOREM 7 *Let G be a finitely presentable group G and let X be a simply connected cellular 2-complex with a free cocompact G action. Then the following statements are equivalent.*

(0) *G is hyperbolic.*

(1) *There exists $K_1 \geqslant 0$ such that, for any $b \in B_1(X, \mathbb{Z})$, $|b|_{fc,\mathbb{Z}} \leqslant K_1 |b|_1$.*

(1') *There exists $K_1' \geqslant 0$ such that for any $b \in B_1(X, \mathbb{Z})$ there exists $a \in C_2(X, \mathbb{Z})$ with $\partial a = b$ and $|a|_1 \leqslant K_1' |b|_1$.*

(2) *There exists $K_2 \geqslant 0$ such that, for any $b \in B_1(X, \mathbb{Z})$, $|b|_{fc,\mathbb{Q}} \leqslant K_2 |b|_1$.*

(3) *There exists $K_3 \geqslant 0$ such that, for any $b \in B_1(X, \mathbb{Q})$, $|b|_{fc,\mathbb{Q}} \leqslant K_3 |b|_1$.*

(3') *There exists $K_3' \geqslant 0$ such that for any $b \in B_1(X, \mathbb{Q})$ there exists $a \in C_2(X, \mathbb{Q})$ with $\partial a = b$ and $|a|_1 \leqslant K_3' |b|_1$.*

(4) *There exists $K_4 \geqslant 0$ such that, for any $b \in B_1(X, \mathbb{R})$, $|b|_{fc,\mathbb{R}} \leqslant K_4 |b|_1$.*

(4') *There exists $K_4' \geqslant 0$ such that for any $b \in B_1(X, \mathbb{R})$ there exists $a \in C_2(X, \mathbb{R})$ with $\partial a = b$ and $|a|_1 \leqslant K_4' |b|_1$.*

(5) *There exists $K_5 \geqslant 0$ such that, for any $b \in B_1(X, \mathbb{R}) \subseteq B_1^{(1)}(X, \mathbb{R})$, $|b|_{f1,\mathbb{R}} \leqslant K_5 |b|_1$.*

(6) *There exists $K_6 \geqslant 0$ such that, for any $b \in B_1^{(1)}(X, \mathbb{R})$, $|b|_{f1,\mathbb{R}} \leqslant K_6 |b|_1$.*

(6') *There exists $K_6' \geqslant 0$ such that, for any $b \in B_1^{(1)}(X, \mathbb{R})$ there exists $a \in C_2^{(1)}(X, \mathbb{R})$ with $\hat{\partial}_2 a = b$ and $|a|_1 \leqslant K_6' |b|_1$.*

(7) *There exists $K_7 \geqslant 0$ such that, for any $z \in Z_1^{(1)}(X, \mathbb{R})$, $|z|_{f1,\mathbb{R}} \leqslant K_7 |z|_1$.*

(7') *There exists $K_7' \geqslant 0$ such that for any $z \in Z_1^{(1)}(X, \mathbb{R})$ there exists $a \in C_2^{(1)}(X, \mathbb{R})$ with $\partial a = z$ and $|a|_1 \leqslant K_7' |z|_1$.*

(8) *There exists* $K_8 \geqslant 0$ *such that, for any* $b \in B_1(X, \mathbb{C})$, $|b|_{fc,\mathbb{C}} \leqslant K_8\,|b|_1$.

(8') *There exists* $K_8' \geqslant 0$ *such that for any* $b \in B_1(X, \mathbb{C})$ *there exists* $a \in C_2(X, \mathbb{C})$ *with* $\partial a = b$ *and* $|a|_1 \leqslant K_8'\,|b|_1$.

(9) *There exists* $K_9 \geqslant 0$ *such that, for any* $b \in B_1(X, \mathbb{C}) \subseteq B_1^{(1)}(X, \mathbb{C})$, $|b|_{f1,\mathbb{C}} \leqslant K_9\,|b|_1$.

(10) *There exists* $K_{10} \geqslant 0$ *such that, for any* $b \in B_1^{(1)}(X, \mathbb{C})$, $|b|_{f1,\mathbb{C}} \leqslant K_{10}\,|b|_1$.

(10') *There exists* $K_{10}' \geqslant 0$ *such that for any* $b \in B_1^{(1)}(X, \mathbb{C})$ *there exists* $a \in C_2^{(1)}(X, \mathbb{C})$ *with* $\hat{\partial}_2 a = b$ *and* $|a|_1 \leqslant K_{10}'\,|b|_1$.

(11) *There exists* $K_{11} \geqslant 0$ *such that, for any* $z \in Z_1^{(1)}(X, \mathbb{C})$, $|z|_{f1,\mathbb{C}} \leqslant K_{11}\,|z|_1$.

(11') *There exists* $K_{11}' \geqslant 0$ *such that for any* $z \in Z_1^{(1)}(X, \mathbb{C})$ *there exists* $a \in C_2^{(1)}(X, \mathbb{C})$ *with* $\partial a = z$ *and* $|a|_1 \leqslant K_{11}'\,|z|_1$.

Remark. Later in the paper only implications (5)⇒(0) and (9)⇒(0) will be used. The author's contribution here is implication (5)⇒(0) and the implications involving complex numbers. This article seems to be the first place where complex numbers are used for isoperimetric inequalities.

Sketch of proof. Equivalences **(1)**⇔**(1′)**, **(3)**⇔**(3′)**, **(4)**⇔**(4′)**, **(6)**⇔**(6′)**, **(7)**⇔**(7′)** **(8)**⇔**(8′)**, **(10)**⇔**(10′)**, **(11)**⇔**(11′)** follow from the definition of the filling norm. (In each case, K' can be taken to be $K + 1$.)
(0)⇒**(1′)** Hyperbolic groups are defined as those having a linear isoperimetric inequality for filling edge-loops with combinatorial disks. Each $b \in B_1(X, \mathbb{Z})$ is a sum of such loops (viewed as chains). Let a be the sum of fillings for these loops. (See [**6**], where the converse is proved.)
(1)⇒**(2)** is obvious because $|\cdot|_{fc,\mathbb{Q}} \leqslant |\cdot|_{fc,\mathbb{Z}}$ on $B_1(X, \mathbb{Z})$. Similarly, **(4)**⇒**(5)**.
(2)⇒**(3)** Let $\Gamma := X^{(1)}$ and $b \in B_1(X, \mathbb{Q})$. Since b is a summable cocycle, being the boundary of a summable 2-chain, it follows b: that Theorem 4 $b = \sum_C g(c)c$ for some $g : C \to [0, \infty)$. The main point here is that the elements of C are *integer* 1-cycles in X. Also, since g is rational, g can be chosen to take rational values (see Theorem 6 for the proof). Fill each $c \in C$ by an integer 2-chain a_c with a linear isoperimetric inequality; then $\sum_C g(c)a_c$ is a rational filling of b with a linear isoperimetric inequality.
(3′)⇒**(4′)** and **(5)**⇒**(7)** are analogous to (2)⇒(3). (See also, [**1**, Proposition 3.6] for the proof of (1)⇔(3)⇔(4).)
(7′)⇒**(6′)**, **(6)**⇒**(5)** are obvious.
(4′)⇒**(8′)** We need to pass from $\mathbb{R}$ to $\mathbb{C}$. If $b \in B_1(X, \mathbb{C})$, then $\operatorname{Re} b$ and $\operatorname{Im} b$ lie in $b \in B_1(X, \mathbb{R})$, so, by (4'), there exist $a, a' \in C_2(X, \mathbb{R})$ with $\partial a = \operatorname{Re} b$, $\partial a' = \operatorname{Im} b$, $|a|_1 \leqslant K_4'\,|\operatorname{Re} b|_1$, $|a'|_1 \leqslant K_4'\,|\operatorname{Im} b|_1$. Then $\partial(a + ia') = b$ and

$$|a + a'|_1 \leqslant K_4'\,|\operatorname{Re} b|_1 + K_4'\,|\operatorname{Im} b|_1 \leqslant 2K_4'\,|b|_1,$$

so we set $K_8' := 2K_4'$.
(8′)⇒**(4′)** We pass from $\mathbb{C}$ to $\mathbb{R}$. Each $b \in B_1(X, \mathbb{R})$ can be viewed as an element of $B_1(X, \mathbb{C})$, therefore, by (8'), there exists $a \in C_2(X, \mathbb{C})$ with with $\partial a = b$ and $|a|_1 \leqslant K_8'\,|b|_1$. Let $\bar{a}$ be the complex conjugate of a. Since b is real, $\partial(\operatorname{Re} a) = \operatorname{Re} b = b$ and $|\operatorname{Re} a|_1 \leqslant |a|_1 \leqslant K_8'\,|b|_1$, so we set $K_4' := K_8'$.

(5)$\Leftrightarrow$**(9)**, **(6′)**$\Leftrightarrow$**(10′)**, **(7′)**$\Leftrightarrow$**(11′)** These implications are similar to (4′)$\Leftrightarrow$(8′).

We devote the rest of this section to the proof of the remaining implication **(5)**$\Rightarrow$**(0)**. It follows immediately from the following proposition (cf. [**3**, Proposition 5.4])

PROPOSITION 8 *Let G and X be as in the hypotheses of Theorem 7 and suppose G is not hyperbolic. Then there exist $C > 0$ and a sequence of geodesic quadrilaterals w in $X^{(1)}$ with*

$$|w|_1 \to \infty \qquad \textit{and} \qquad |w|_{f1,\mathbb{R}} \geqslant C(|w|_1)^2.$$

Here, abusing notation, we identify the quadrilateral w and the corresponding integral 1-chain.

Proof. We modify the techniques of [**2**], [**3**] which in turn used a modification of Ol'shanskii's method of layers [**12**].

Each 2-cell in X is a polygon whose sides are glued to edges in $X^{(1)}$. Let M be the maximal number of sides over all 2-cells in X. In particular, all combinatorial boundaries of 2-cells in X have diameter at most M, and also, for any 2-chain a in X,

$$|\partial a|_1 \leqslant M|a|_1. \tag{4}$$

If G is not hyperbolic, Ol'shanskii shows that there are arbitrarily *thick* geodesic quadrilaterals in $X^{(1)}$, and Gersten states more precisely that (see [**3**, Proposition 5.2] in which we make the substitution $t = 8Mr$) there exist

- a sequence of integers r tending to infinity,
- a geodesic quadrilateral $w = w(r)$ for each r,
- a (geodesic) side S in each w, and
- a subinterval $[x, y] \subseteq S$ with the midpoint z

such that

- the length of $[x, y]$ is $4Mr$,
- the distance from z to the sides of w other than S is at least $4Mr$, and
- the perimeter of w is at most $80Mr$.

This last property implies that

$$|w|_1 \leqslant 80Mr. \tag{5}$$

Since the interval $[x, y]$ does not intersect other sides of the quadrilateral,

$$|w|_1 \geqslant 4Mr \to \infty \quad \text{as} \quad r \to \infty.$$

For each positive integer $k \leqslant r$, let B_k be the simplicial ball of radius $4Mk$ centred at z. If a is any summable chain with $\partial a = w$, denote by a_k the restriction of a to (the 2-cells of) $B_k \setminus B_{k-1}$ (with $a_k = 0$ outside). Let s_k and t_k be the two points on $[x, y]$ which are at distance $4Mk - 2M$ from z (see Fig. 1), and

$$A_k := \{v \in [x, y] \mid 4M(k-1) \leqslant d(z, v) \leqslant 4Mk - 2M\},$$
$$A'_k := \{v \in [x, y] \mid 4Mk - 2M \leqslant d(z, v) \leqslant 4Mk\}.$$

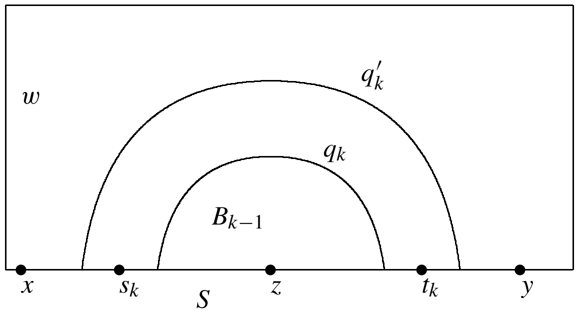

Fig. 1 The filling a of b.

Here ∂a_k is a 1-cycle which takes the value 1 on the edges of $[x, y]$ incident on s_k and t_k (since these edges are 'deep inside' $B_k \setminus B_{k-1}$). Also, ∂a_k splits as the sum of a 1-chain q_k supported in

$$\{M - \text{neighbourhood of } B_{k-1}\} \cup A_k,$$

and a 1-chain q'_k supported in

$$\{M - \text{neighbourhood of } \overline{X \setminus B_k}\} \cup A'_k.$$

These two supports intersect only in $T := \{s_k, t_k : 0 < k \leqslant r\}$, and the above observation implies that

$$\partial q_k = t_k - s_k \qquad \text{and} \qquad \partial q'_k = s_k - t_k. \tag{6}$$

By Theorem 6, each q_k splits as a (finite) linear combination of T-paths

$$q_k = \sum_i \alpha_i p_i$$

such that

$$\alpha_i \geqslant 0 \qquad \text{and} \qquad |q_k|_1 = \sum_i \alpha_i |p_i|_1.$$

Those T-paths p_j which have non-trivial boundary must be of length at least $d(s_k, t_k) \geqslant 8Mk - 4M$, and (6) says that $\sum_j \alpha_j \geqslant 1$, so

$$|q_k|_1 = \sum_i \alpha_i |p_i|_1 \geqslant \sum_j \alpha_j (8Mk - 4M) \geqslant 8Mk - 4M.$$

The same holds for q'_k, hence

$$|\partial a_k|_1 = |q_k|_1 + |q'_k|_1 \geqslant 2(8Mk - 4M) = 16Mk - 4M.$$

By (4),

$$|a_k|_1 \geqslant \frac{1}{M}|\partial a_k|_1 \geqslant 16k - 4,$$

and since the chains a_k have disjoint supports,

$$|a|_1 \geqslant \sum_{k=1}^{r} |a_k|_1 \geqslant \sum_{k=1}^{r}(16k-4) = 16\frac{r(r+1)}{2} - 4r \geqslant 8r^2.$$

Then, by (5),

$$|a|_1 \geqslant 8\left(\frac{|w|_1}{80M}\right)^2 = \frac{1}{800M^2} \cdot (|w|_1)^2.$$

Since this is true for any summable filling a of w, it follows that

$$|w|_{f1,\mathbb{R}} \geqslant \frac{1}{800M^2} \cdot (|w|_1)^2,$$

so we put $C := \frac{1}{800\ \mathrm{M}^2}$. Proposition 8 and Theorem 7 are proved.

5. The characterization

In this section we characterize hyperbolic groups by bounded cohomology.

Let G be a finitely presentable group and let $\mathcal{M}(G)$ be one of the following ten classes of modules.

- $\mathcal{M}_1(G)$, the class of *bounded* $\mathbb{Q}G$*-modules*. A $\mathbb{Q}G$-module is called *bounded* if it is normed as a vector space over $\mathbb{Q}$ and G acts on it by linear operators of uniformly bounded norms.
- $\mathcal{M}_2(G)$, the class of *isometric* $\mathbb{Q}G$*-modules*. A $\mathbb{Q}G$-module is called *isometric* if it is bounded and, moreover, the G-action preserves its norm.
- $\mathcal{M}_3(G)$, analogously, the class of *bounded* $\mathbb{R}G$*-modules*.
- $\mathcal{M}_4(G)$, analogously, the class of *isometric* $\mathbb{R}G$*-modules*.
- $\mathcal{M}_5(G)$, analogously, the class of *bounded Banach* $\mathbb{R}G$*-modules*, that is, bounded $\mathbb{R}G$-modules which are Banach spaces with respect to their norms.
- $\mathcal{M}_6(G)$, analogously, the class of *isometric Banach* $\mathbb{R}G$*-modules*.
- $\mathcal{M}_7(G)$, the class of *bounded* $\mathbb{C}G$*-modules*.
- $\mathcal{M}_8(G)$, the class of *isometric* $\mathbb{C}G$*-modules*.
- $\mathcal{M}_9(G)$, analogously, the class of *bounded Banach* $\mathbb{C}G$*-modules*.
- $\mathcal{M}_{10}(G)$, the class of *isometric Banach* $\mathbb{C}G$*-modules*.

Obviously, $\mathcal{M}_{10}(G) \subseteq \mathcal{M}(G) \subseteq \mathcal{M}_1(G)$ for each such $\mathcal{M}(G)$.

As promised in the introduction, Theorem 3 is stated more precisely as follows.

THEOREM 9 *Let G be a finitely presentable group, and $\mathcal{M}(G)$ one of the classes $\mathcal{M}_1(G)$ to $\mathcal{M}_{10}(G)$ described above. Then the following statements are equivalent.*

(a) *G is hyperbolic.*

(b) *The map* $H_b^2(G, V) \to H^2(G, V)$ *is surjective for any* $V \in \mathcal{M}(G)$.

(c) *The map* $H_b^i(G, V) \to H^i(G, V)$ *is surjective for any* $i \geqslant 2$ *and any* $V \in \mathcal{M}(G)$.

Proof. $(a) \Rightarrow (c)$ was shown in [**9**] for the largest class $\mathcal{M}_1(G)$. Property (c) for $\mathcal{M}_1(G)$ implies the same property for each of the classes $\mathcal{M}_1(G)$ to $\mathcal{M}_{10}(G)$. The implication $(c) \Rightarrow (b)$ is obvious. It only remains to prove $(b) \Rightarrow (a)$ for the class $\mathcal{M}_{10}(G)$. Below we present the proof of $(b) \Rightarrow (a)$ for class $\mathcal{M}_6(G)$, since real coefficients are often used. To show $(b) \Rightarrow (a)$ for class $\mathcal{M}_{10}(G)$ the reader would just need to replace $\mathbb{R}$ by $\mathbb{C}$ everywhere.

5.1. *Proof of* $(b) \Rightarrow (a)$ *for class* $\mathcal{M}_6(G)$

Since G is finitely presentable, there exists a contractible cell complex X with a free cellular G-action such that the induced action on the 2-skeleton of X is cocompact. We take V to be $B_1^{(1)}(X, \mathbb{R})$ with the filling norm $|\cdot|_{f1,\mathbb{R}}$ which we denote for simplicity by $|\cdot|_f$. The vector space V is Banach as the quotient of the Banach space $C_2^{(1)}(X, \mathbb{R})$ by the (closed) kernel of $\hat{\partial}_2 : C_2^{(1)}(X, \mathbb{R}) \to C_1^{(1)}(X, \mathbb{R})$. In particular, $V \in \mathcal{M}_6(G)$.

Let Y be the geometric realization of the homogeneous bar-construction for G, that is, Y is the simplicial complex whose k-simplices are labelled by ordered $(k+1)$-tuples $[x_0, \dots, x_k]$ of elements of the group G, and each simplex labeled $[x_0, \dots, \widehat{x_i}, \dots, x_k]$ is identified with the ith face of $[x_0, \dots, x_k]$. The action of G on Y is diagonal:

$$g \cdot [x_0, \dots, x_k] := [g \cdot x_0, \dots, g \cdot x_k].$$

Let $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ be the augmented chain complexes

$$\dots \longrightarrow C_2(X, \mathbb{R}) \xrightarrow{\partial_2} C_1(X, \mathbb{R}) \xrightarrow{\partial_1} C_0(X, \mathbb{R}) \xrightarrow{\epsilon} \mathbb{R} \longrightarrow 0$$

and

$$\dots \longrightarrow C_2(Y, \mathbb{R}) \xrightarrow{\partial_2} C_1(Y, \mathbb{R}) \xrightarrow{\partial_1} C_0(Y, \mathbb{R}) \xrightarrow{\epsilon} \mathbb{R} \longrightarrow 0\ ,$$

respectively. Both X and Y are contractible, hence $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ are acyclic. Both $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ have free $\mathbb{R}G$-modules in each non-negative dimension. Also $\mathcal{C}_i^X$ is finitely generated as an $\mathbb{R}G$-module for $i = 0, 1, 2$.

Obviously, $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ coincide in the negative dimensions. By the standard uniqueness of resolutions property, there exist homotopy equivalences $\varphi_* : \mathcal{C}_*^Y \to \mathcal{C}_*^X$ and $\psi_* : \mathcal{C}_*^X \to \mathcal{C}_*^Y$ which are identities in dimension -1. It is important that φ_* and ψ_* are chain maps in the category of $\mathbb{R}G$-modules, in particular, for each i, φ_i and ψ_i are linear maps commuting with the G-action.

For the dual cochain complexes

$$\mathcal{C}_X^i := C^*(X, V) = \mathrm{Hom}_{\mathbb{R}G}(\mathcal{C}_i^Y, V) \quad \text{and} \quad \mathcal{C}_Y^i := C^*(Y, V) = \mathrm{Hom}_{\mathbb{R}G}(\mathcal{C}_i^Y, V)$$

and the dual maps

$$\varphi^* : \mathcal{C}_X^* \to \mathcal{C}_Y^* \quad \text{and} \quad \psi^* : \mathcal{C}_Y^* \to \mathcal{C}_X^*,$$

the cochain map $\psi^* \circ \varphi^*$ is homotopic to the identity map, hence $\psi^* \circ \varphi^*$ induces the identity map on cohomology $H^*(G, V)$ in the positive dimensions.

The *universal cocycle* in $\mathcal{C}_X^2$ is the 2-cochain $u : \mathcal{C}_2^X \to V$ which coincides with the composition

$$C_2(X, \mathbb{R}) \xrightarrow{\partial_2} B_1(X, \mathbb{R}) \hookrightarrow B_1^{(1)}(X, \mathbb{R}).$$

One checks that u is indeed a cocycle. By the above observations,

$$u = (\psi^2 \circ \varphi^2)(u) + \delta v \tag{7}$$

for some 1-cochain $v : \mathcal{C}_1^X \to V$.

Since $\varphi^2(u)$ is a cocycle in $\mathcal{C}_Y^2$ and the map $H_b^2(G, V) \to H^2(G, V)$ is surjective by the assumption,

$$\varphi^2(u) = u' + \delta v' \tag{8}$$

for a 1-cochain $v' \in \mathcal{C}_Y^1$ and a *bounded* 2-cocycle $u' \in \mathcal{C}_Y^2$, that is,

$$|u'|_\infty < \infty. \tag{9}$$

The above information is demonstrated by the diagrams

$$\begin{array}{ccc} a, b \in \mathcal{C}_*^X & = & C_*(X, \mathbb{R}) \\ {\scriptstyle\varphi_*}\big\uparrow\big\downarrow{\scriptstyle\psi_*} & & \\ \mathcal{C}_*^Y & = & C_*(Y, \mathbb{R}) \end{array} \quad \text{and} \quad \begin{array}{ccc} u, v \in \mathcal{C}_X^* & = & C^*(X, V) \\ {\scriptstyle\varphi^*}\big\downarrow\big\uparrow{\scriptstyle\psi^*} & & \\ u', v' \in \mathcal{C}_Y^* & = & C^*(Y, V). \end{array}$$

We also have the standard pairings $\langle \cdot, \cdot \rangle : \mathcal{C}_X^i \oplus \mathcal{C}_i^X \to V$ and $\langle \cdot, \cdot \rangle : \mathcal{C}_Y^i \oplus \mathcal{C}_i^Y \to V$.

Fix any vertex y in Y (that is, $y \in G$). For each 1-chain $b = \sum_{y_0, y_1 \in G} \beta_{[y_0, y_1]}[y_0, y_1]$ in Y, *the cone* over b with vertex y is the 2-chain

$$[y, b] := \sum_{y_0, y_1 \in G} \beta_{[y_0, y_1]}[y, y_0, y_1].$$

If b happens to be a cycle, then $\partial[y, b] = b$. Obviously,

$$\Big|[y, b]\Big|_1 = |b|_1.$$

In other words, $[y, b]$ is a filling of b in Y, and the above formula says that these fillings satisfy a linear isoperimetric inequality in Y. We want the same property in X to show hyperbolicity, so we will use the maps φ^* and ψ^* to 'transfer' the linear isoperimetric inequality from Y to X using the fact that the universal cocycle is cohomologous to a bounded cocycle.

Note also that if α is a cocycle in $\mathcal{C}_Y^i$ and $c \in \mathcal{C}_i^Y$, then $c - [y, \partial c]$ is a cycle, hence a boundary, so $\langle \alpha, c - [y, \partial c] \rangle = 0$ and

$$\langle \alpha, c \rangle = \langle \alpha, [y, \partial c] \rangle. \tag{10}$$

Now with the above setup we can finish the proof of Theorem 9. Pick any 1-boundary $b \in B_1(X, \mathbb{R})$ and any 2-chain a with $\partial a = b$. We will show that $|b|_f \leqslant K|b|_1$ for some uniform constant K, therefore G will be hyperbolic by implication (5)⇒(0) in Theorem 7. (By implication (9)⇒(0) in case of complex coefficients.)

By equality (7),

$$b = \partial a = \langle u, a \rangle = \langle (\psi^2 \circ \varphi^2)(u) + \delta v, a \rangle = \langle (\psi^2 \circ \varphi^2)(u), a \rangle + \langle v, b \rangle.$$

Since $\varphi^2(u)$ is a cocycle, and using (10) and (8),

$$\begin{aligned}&\big\langle(\psi^2\circ\varphi^2)(u),a\big\rangle=\big\langle\varphi^2(u),\psi_2(a)\big\rangle=\big\langle\varphi^2(u),[y,\partial(\psi_2(a))]\big\rangle\\&=\big\langle\varphi^2(u),[y,\psi_1(b)]\big\rangle=\big\langle u'+\delta v',[y,\psi_1(b)]\big\rangle=\big\langle u',[y,\psi_1(b)]\big\rangle+\big\langle v',\partial[y,\psi_1(b)]\big\rangle\\&=\big\langle u',[y,\psi_1(b)]\big\rangle+\big\langle v',\psi_1(b)\big\rangle=\big\langle u',[y,\psi_1(b)]\big\rangle+\big\langle\psi^1(v'),b\big\rangle.\end{aligned}$$

So, combining the above two formulae,

$$\begin{aligned}b&=\big\langle u',[y,\psi_1(b)]\big\rangle+\big\langle\psi^1(v')+v,b\big\rangle,\\|b|_f&\leqslant\Big|\big\langle u',[y,\psi_1(b)]\big\rangle\Big|_f+\Big|\big\langle\psi^1(v')+v,b\big\rangle\Big|_f\\&\leqslant|u'|_\infty\cdot\Big|[y,\psi_1(b)]\Big|_1+\Big|\psi^1(v')+v\Big|_\infty\cdot|b|_1\\&=|u'|_\infty\cdot\big|\psi_1(b)\big|_1+\Big|\psi^1(v')+v\Big|_\infty\cdot|b|_1\\&\leqslant\Big(|u'|_\infty\cdot|\psi_1|_\infty+\Big|\psi^1(v')+v\Big|_\infty\Big)\cdot|b|_1.\end{aligned}$$

This will give the desired linear isoperimetric inequality once we prove that all the norms in the parentheses are finite.

The cochain u' is bounded by definition (by a constant depending only on the choice of G and X, see (9)). The maps $\psi_1:\mathcal{C}_1^X\to\mathcal{C}_1^Y$ and $\psi^1(v')+v:\mathcal{C}_1^X\to V$ are both linear maps commuting with the G-action. Their boundedness (by constants depending only on G and X) is immediate from the following simple observation which deserves the status of a lemma.

LEMMA 10 *Let $(W,|\cdot|_1)$ and $(W',|\cdot|)$ be two normed vector spaces over $\mathbb{R}$, where $|\cdot|_1$ is the ℓ_1-norm on W with respect to some basis. Suppose a group G acts on both W and W' such that*

- *on W it permutes the basis so that there are only finitely many orbits of basis elements, and*
- *on W' it preserves the norm $|\cdot|$.*

Then, if $f:W\to W'$ is a linear map commuting with the G-action, then f is bounded, that is, $|f|_\infty<\infty$.

Proof. If w_1 and w_2 are two basis elements of W in the same G-orbit, that is, $w_1=g\cdot w_2$, then

$$|f(w_1)|=|f(g\cdot w_2)|=|g\cdot f(w_2)|=|f(w_2)|.$$

Since there are only finitely many G-orbits of basis elements in W, $|f(\cdot)|$ takes only finitely many values on the basis elements, hence $|f|_\infty<\infty$. This proves Lemma 10 and Theorem 9.

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